EULER LINES, TRITANGENT CENTERS, AND THEIR TRIANGLES

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1. Introduction. Notations and Early History. In the Euclidean geometry of the triangle, it has been known since ancient times that the medians concur, and so do the altitudes, the perpendicular bisectors of the sides, and the internal angle bisectors. Let us call the triangle ABC, and name these points of concurrence as centroid G, orthocenter H, circumcenter O, and incenter I_0, respectively. Let I_1, I_2, I_3 be the excenters opposite the angles A, B, C, respectively. A point which is either the incenter or one of the excenters of a triangle is called a "tritangent center," and upon occasion will be denoted by I.

The first systematic study of mutual relationships between these centers was made in the eighteenth century by Euler [**8**], who showed that O, G, H are collinear, with G dividing the segment OH in the ratio $1:2$. He also found a number of expressions for the distances between these centers. Early in the nineteenth century Brianchon and Poncelet [**7**], [**16**] discussed the nine-point circle whose center N is at the midpoint of OH. Then shortly thereafter Feuerbach [**9**], [**16**] proved that the nine-point circle touches all four tritangent circles.

In all these developments the triangle ABC was regarded as given and the properties of the centers were investigated. In the present note this process is reversed; the points O, G, H on the Euler line are regarded as given and it is investigated how the position of any one tritangent center I can serve to determine the triangle. This quickly leads to the discovery that the incenter must always lie inside the circle on GH as diameter, and that all three excenters must lie outside it. This suggests the following nomenclature:

DEFINITION 1. For nonequilateral triangles ABC the "critical circle" shall mean the circle on GH as diameter.

Furthermore, outside the critical circle there is a region, bounded by a closed bicircular quartic curve, inside which there cannot lie any tritangent center. We shall accordingly call this region the "acentric lacuna." Its precise definition must be deferred; it is given in §8.

If due allowances are made for degenerate and limiting cases, then the resulting picture is complete in the sense that all points not in the acentric lacuna are possible positions of a tritangent center I of a triangle with the given Euler line OGH. As a result the plane can be partitioned into regions corresponding to the type of tritangent center therein and the shape of the triangle ABC. The boundaries of these regions are formed by a family of bicircular quartics. These quartics can be generated as inverses of conics, but that is beyond the scope of the present note.

Since the Euler line of an equilateral triangle is indeterminate, it will be assumed initially that ABC is not equilateral. The equilateral triangle as a limiting case is discussed in §7.

2. Preliminary Results. Let R be the circumradius, r the inradius, and r_1, r_2, r_3 the radii of the excircles about the excenters I_1, I_2, I_3. Then elementary trigonometric results are [**12**]:

$$(1)\qquad r = 4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C = R(\cos A + \cos B + \cos C - 1),$$

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(2) $$r_1 = 4R\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C = R(\cos B + \cos C - \cos A + 1),$$

(3) $$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C,$$

(4) $$OI_0^2 = R(R - 2r),$$

(5) $$OI_1^2 = R(R + 2r_1),$$

(6) $$I_0N = \frac{1}{2}R - r,$$

(7) $$I_1N = \frac{1}{2}R + r_1,$$

(8) $$OH^2 = R^2(1 - 8\cos A\cos B\cos C).$$

Of these (4), (5), and (8) are usually attributed to Euler [**5**], [**8**] though Mackay [**14**] cites several earlier forms of (4) in short-lived English journals of the mideighteenth century. Formulas (6) and (7) are equivalent to Feuerbach's theorem [**12**], [**16**].

3. The Critical Circle. By considering angles subtended at vertices it is easily shown that the circumcenter O and the incenter I_0 cannot coincide if ABC is not equilateral. Hence by (4) and (6)

$$OI_0^2 - 4 \times I_0N^2 = R(R - 2r) - (R - 2r)^2 = 2r(R - 2r) = \frac{2r}{R} \times OI_0^2 > 0,$$

since $OI_0 > 0$. Hence

(9) $$OI_0 > 2 \times I_0N.$$

Similarly from (5) and (7)

$$OI_1^2 - 4 \times I_1N^2 = R(R + 2r_1) - (R + 2r_1)^2 = -2r_1(R + 2r_1) < 0,$$

whence

(10) $$OI_1 < 2 \times I_1N.$$

Now the locus of points P for which $OP = 2 \times PN$ is a circle of Appollonius [**1**]. Since $OG = 2 \times GN$ and $OH = 2 \times HN$, this circle has diameter GH, and thus it is the critical circle of Definition 1. Further P lies inside or outside this circle according as $OP \gtrless 2 \times PN$, so we have:

THEOREM 1. *The incenter of a nonequilateral triangle lies inside the critical circle and all the excenters lie outside it.*

4. The Cubic Equation for Angle Cosines. Suppose that the position of the incenter I_0 relative to the Euler line is given, so that OI_0, I_0N, and OH are known quantities. Then (1), (3), (4), (6), and (8) can be used to construct a cubic whose roots are the cosines of the angles of ABC. Let $OI_0 = \rho$, $I_0N = \sigma$, $OH = \kappa$, and put

$$\alpha = 2\sigma^2/3\rho^2, \quad \beta = \kappa^2/2\rho^2.$$

Then by (4) and (6)

(11) $$R = OI_0^2/2 \times I_0N = \rho^2/2\sigma,$$

and by (8)

(12) $$\cos A\cos B\cos C = \frac{1}{8}\left(1 - \frac{\kappa^2}{R^2}\right) = \frac{1}{8}\left(1 - \frac{4\kappa^2\sigma^2}{\rho^4}\right) = \frac{1}{8}(1 - 12\alpha\beta).$$

Hence by (3)

$$\cos^2 A + \cos^2 B + \cos^2 C = \frac{3}{4} + 3\alpha\beta = \frac{3}{4}(1 + 4\alpha\beta). \tag{13}$$

By (6) and (11)

$$\frac{r}{R} = \frac{\frac{1}{2}R - \sigma}{R} = \frac{1}{2} - \frac{2\sigma^2}{\rho^2} = \frac{1}{2}(1 - 6\alpha). \tag{14}$$

Then by (1)

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = \frac{3}{2}(1 - 2\alpha), \tag{15}$$

and by (13) and (15)

$$\begin{aligned}
&\cos B \cos C + \cos C \cos A + \cos A \cos B \\
&= \frac{1}{2}(\cos A + \cos B + \cos C)^2 - \frac{1}{2}(\cos^2 A + \cos^2 B + \cos^2 C) \\
&= \frac{9}{8}(1 - 2\alpha)^2 - \frac{3}{8}(1 + 4\alpha\beta) \\
&= \frac{3}{4}(6\alpha^2 - 2\alpha\beta - 6\alpha + 1).
\end{aligned} \tag{16}$$

Hence by (12), (15), and (16) the cubic in c

$$c^3 + \frac{3}{2}(2\alpha - 1)c^2 + \frac{3}{4}(6\alpha^2 - 2\alpha\beta - 6\alpha + 1)c + \frac{1}{8}(12\alpha\beta - 1) = 0 \tag{17}$$

has roots $c = \cos A, \cos B, \cos C$. Expressed in terms of ρ, σ, κ it can be written

$$\rho^4(1 - 2c)^3 + 8\rho^2\sigma^2 c(3 - 2c) - 16\sigma^4 c - 4\sigma^2\kappa^2(1 - c) = 0.$$

Applying a similar argument ot the excenter I_1, we can put $OI_1 = \rho_1$, $I_1N = \sigma_1$, and

$$\alpha_1 = 2\sigma_1^2/3\rho_1^2, \quad \beta_1 = \kappa^2/2\rho_1^2. \tag{18}$$

Then, successively,

$$\begin{aligned}
R &= OI_1^2/2 \times I_1N = \rho_1^2/2\sigma_1, \\
&\cos A \cos B \cos C = \tfrac{1}{8}(1 - 12\alpha_1\beta_1), \\
&\cos^2 A + \cos^2 B + \cos^2 C = \tfrac{3}{4}(1 + 4\alpha_1\beta_1), \\
\frac{r_1}{R} &= \frac{\sigma_1 - \frac{1}{2}R}{R} = \frac{2\sigma_1^2}{\rho_1^2} - \frac{1}{2} = \frac{1}{2}(6\alpha_1 - 1),
\end{aligned} \tag{19}$$

$$\cos A - \cos B - \cos C = 1 - \frac{r_1}{R} = \frac{3}{2}(1 - 2\alpha_1), \tag{20}$$

$$\begin{aligned}
&\cos B \cos C - \cos C \cos A - \cos A \cos B \\
&= \frac{1}{2}(\cos A - \cos B - \cos C)^2 - \frac{1}{2}(\cos^2 A + \cos^2 B + \cos^2 C) \\
&= \tfrac{9}{8}(1 - 2\alpha_1)^2 - \tfrac{3}{8}(1 + 4\alpha_1\beta_1) = \tfrac{3}{4}(6\alpha_1^2 - 2\alpha_1\beta_1 - 6\alpha_1 + 1).
\end{aligned} \tag{21}$$

Equations (19), (20), (21) are of the same form as (12), (15), (16) but with the signs of $\cos B$ and $\cos C$ reversed. Hence the cubic in c

$$c^3 + \frac{3}{2}(2\alpha_1 - 1)c^2 + \frac{3}{4}(6\alpha_1^2 - 2\alpha_1\beta_1 - 6\alpha_1 + 1)c + \frac{1}{8}(12\alpha_1\beta_1 - 1) = 0 \tag{22}$$

has roots

$$c = \cos A,\ -\cos B,\ -\cos C$$
$$= \cos A, \cos(\pi - B), \cos(\pi - C).$$

If $\theta_1, \theta_2, \theta_3$ are the inverse cosines of the roots of the cubic (22), then

$$\theta_1 + \theta_2 + \theta_3 = A + (\pi - B) + (\pi - C) = \pi + 2A.$$

Thus, if the angles $\theta_1, \theta_2, \theta_3$ are found by means of the cubic (22), then the angle opposite the given excenter can be identified as $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - \pi)$, and this will be equal to one of θ_1, θ_2, or θ_3. The other angles of the triangle are then equal to the supplements of the other two thetas.

Since the cubics (17) and (22) are of the same form, the results for incenters and excenters can be combined by dropping suffixes, thus:

THEOREM 2. *If I is a tritangent center of ABC and $OI = \rho$, $IN = \sigma$, $OH = \kappa$, then the roots of the cubic in c*

$$\rho^4(1 - 2c)^3 + 8\rho^2\sigma^2 c(3 - 2c) - 16\sigma^4 c - 4\sigma^2\kappa^2(1 - c) = 0 \tag{23}$$

are cosines of the angles of the triangle or of their supplements. If $\cos\theta_1, \cos\theta_2, \cos\theta_3 : (0 < \theta_1, \theta_2, \theta_3 < \pi)$ *are the roots of this cubic, and*

(i) $\theta_1 + \theta_2 + \theta_3 = \pi$, *then I is the incenter and the angles of the triangle are $\theta_1, \theta_2, \theta_3$;*
(ii) $\theta_1 + \theta_2 + \theta_3 \neq \pi$, *then I is an excenter and the angle opposite it is $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - \pi)$. Further, this is equal to one of $\theta_1, \theta_2, \theta_3$ and the remaining two are supplements of the remaining angles of the triangle.*

5. Loci of Tritangent Centers of Isosceles Triangles. Triage. If ABC is isosceles, with $AB = AC$, then the Euler line is the bisector of the angle at A. Hence both the incenter and the excenter opposite A lie on the Euler line.

The projections of AB and AH onto AC are identical, so

$$AB\cos A = 2R\sin C\cos A = AH\sin C, \text{ whence } AH = 2R\cos A.$$

For $0 < A < \pi/3$, as in Fig. 1, $AH = 2R\cos A > R = AO$, so

$$\kappa = OH = R(2\cos A - 1).$$

Similarly, if $\pi/3 < A < \pi$, then

$$\kappa = R(1 - 2\cos A).$$

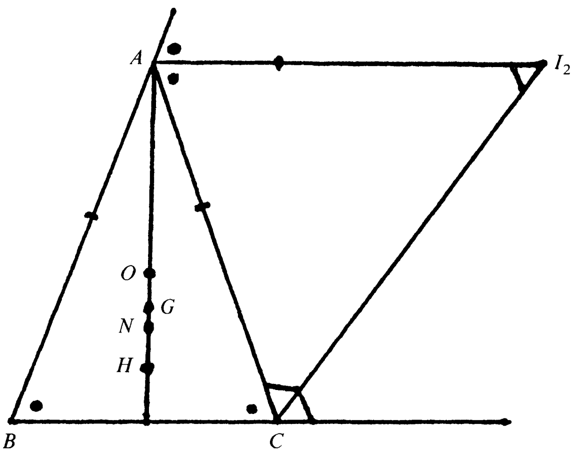

FIG. 1.

In each case AI_2 is parallel to BC so the triangle ACI_2 is isosceles, and $AI_2 = AC = 2R\sin B$. Also

$$\cos A = \cos(\pi - 2B) = 2\sin^2 B - 1.$$

Now let $\rho = OI_2$, $\sigma = I_2N$ so that (ρ, σ) can be regarded as bipolar coordinates of I_2 with respect to poles O and N [**13**]. Then

$$\rho^2 = OI_2^2 = AO^2 + AI_2^2 = R^2 + 4R^2\sin^2 B = R^2(2\cos A + 3),$$

$$AN = R \pm \frac{1}{2}\kappa = \frac{1}{2}R(2\cos A + 1) \text{ according as } A \lessgtr \pi/3,$$

$$\sigma^2 = I_2N^2 = AN^2 + AI_2^2 = \frac{1}{4}R^2(2\cos A + 3)^2.$$

Hence $\sigma = \frac{1}{2}R(2\cos A + 3)$ and

$$2(\sigma^2 - \rho^2) = \frac{1}{2}R^2\{(2\cos A + 3)^2 - 4(2\cos A + 3)\}$$
$$= \frac{1}{2}R^2(2\cos A - 1)(2\cos A + 3) = \pm\kappa\sigma.$$

Squaring, it follows that

$$4(\sigma^2 - \rho^2)^2 = \kappa^2\sigma^2 \tag{24}$$

is the bipolar equation of the locus of I_2 relative to the Euler line OH.

If O is taken as origin of cartesian coordinates x, y, and H as unit point $(1,0)$ on the x-axis, then

$$\rho^2 = x^2 + y^2, \quad \sigma^2 = \left(x - \frac{1}{2}\right)^2 + y^2, \quad \kappa = 1. \tag{25}$$

The cartesian equivalent of (24) then becomes

$$y^2 = 3x^2 - x.$$

This is a hyperbola of eccentricity 2 whose major axis is the Euler line. One branch touches the critical circle at G and the other branch goes through O. The excenter I_2 lies on the first branch or the second according as the apex angle A of the isosceles triangle is greater than or less than 60°.

The hyperbola divides the plane into three regions. In the region containing the negative x-axis the point $(-1 - \sqrt{2}, 0)$ is the excenter opposite the right angle for an isosceles right-angled triangle. By continuity all other tritangent centers in this region are also excenters opposite the largest angle of their triangles. Similarly a tritangent center in the region between the two branches is an excenter opposite the intermediate angle of its triangle. The remaining region contains the critical circle, so an excenter opposite the least angle of its triangle must lie inside the region but outside the critical circle.

A classification of elements into three disjoint sets is sometimes called a "triage." Hence:

DEFINITION 2. For a given Euler line the "triage-hyperbola" shall mean the hyperbola defined by the bipolar equation

$$4(\sigma^2 - \rho^2)^2 = \kappa^2\sigma^2.$$

6. Quartic Loci of I for Triangles with One Fixed Angle. If one angle of ABC is equal to θ, then the bipolar coordinates ρ, σ of both the incenter and the excenter opposite that angle must satisfy (23) with $c = \cos\theta$. If we substitute (25) and rearrange, it follows that the cartesian coordinates of these centers satisfy

$$(26)\quad \begin{aligned}&(2\cos\theta-1)(2\cos\theta+1)^2(x^2+y^2)^2-8\cos\theta(2\cos\theta+1)x(x^2+y^2)\\&+2(2\cos^2\theta-\cos\theta+2)(x^2+y^2)+16(\cos\theta)x^2-4(\cos\theta+1)x+1=0.\end{aligned}$$

In general, this is the equation of a bicircular quartic symmetric about the x-axis. Let us call the curve $Q(\theta)$. Particular cases of note are:

$Q(0)$: $\{(x-\frac{2}{3})^2+y^2-\frac{1}{9}\}^2=0$,

the critical circle, repeated;

$Q(\pi/3)$: $(x-\frac{1}{2})\{(x-\frac{1}{2})^2+y^2\}=0$,

the line perpendicular to the Euler line at N, and a point circle at N [**12**];

$Q(\frac{1}{2}\pi)$: $(x^2+y^2-2)^2=5-4x$,

the limaçon of Pascal with node at the orthocentre H [**13**];

$Q(2\pi/3)$: $2x^2-6y^2+2x-1=0$,

a hyperbola of eccentricity $2/\sqrt{3}$;

(27) $Q(\pi)$: $3(x^2+y^2)^2+(8x-10)(x^2+y^2)+16x^2-1=0$,

a closed curve touching the critical circle and the triage-hyperbola at G. It has a simple cusp at $(-1,0)$ and lies entirely between its two bitangents $x=\frac{1}{2}$ and $x=-7/4$.

DEFINITION 3. For a given Euler line the "extremal quartic" shall mean the bicircular quartic defined by equation (27).

For given values of θ and x the equation (26) is a quadratic in y^2, so it is easy to compute coordinates and plot $Q(\theta)$. For incenters the only relevant parts of the curves are those inside the critical circle.

7. Limiting and Degenerate Cases. Not all locations of I relative to the Euler line correspond to real, finite, nondegenerate triangles.

First, if I approaches the nine-point center N, then by (11) the circumradius R tends to infinity and the cubic (23) approaches the form $(1-2c)^3=0$. That is, if $\kappa=OH$ is kept constant, then N must be interpreted as the limiting position of incenters of near-equiangular triangles of increasing size.

Next, if I is on $Q(0)$, the repeated critical circle, then at least one root of (23) is $c=1=\cos 0$, so the triangle is degenerate, with at least one zero angle. If the zero angle is A, then B, C, I_0, I_1 all coincide with I.

If I is on the extremal quartic $Q(\pi)$, then at least one root of (23) is $c=-1=\cos\pi$ and again the triangle is degenerate, with a zero angle; but I is now an excenter not opposite that zero angle.

If I is at G, where $Q(0)$ and $Q(\pi)$ touch, then the triangle must be regarded as doubly degenerate, with angles $0, 0, \pi$, circumradius $\kappa/3$, but all sides zero.

8. The Acentric Lacuna. A real triangle corresponds to a location of I relative to the Euler line if and only if the resulting cubic (23) has all its roots real and in the range $-1\leqslant c\leqslant 1$. If $z=2c+2\alpha-1$ is substituted in (17), then the cubic for z becomes

$$z^3+6\alpha(\alpha-\beta-1)z-2\alpha(10\alpha^2-6\alpha\beta-15\alpha-3\beta+6)=0.$$

The discriminant of the latter cubic is [**17**]

$$\Delta = 8\alpha^3(\alpha - \beta - 1)^3 + \alpha^2(10\alpha^2 - 6\alpha\beta - 15\alpha - 3\beta + 6)^2 \tag{28}$$
$$= \alpha^2\{(3\alpha - \beta + 2)^2 - 24\alpha\}\{12\alpha^2 - 8\alpha\beta - 20\alpha + 9\},$$

and the roots are all real if and only if $\Delta \leqslant 0$.

Expressed in terms of ρ, σ, κ, the second factor becomes

$$\{(4\rho^2 + 4\sigma^2 - \kappa^2)^2 - 64\rho^2\sigma^2\}/4\rho^4. \tag{29}$$

Now $\rho, \sigma, \frac{1}{2}\kappa$ are sides of the triangle ONI, so they are nonnegative and $\rho + \sigma \geqslant \frac{1}{2}\kappa$ with equality only if I lies on ON. Hence

$$\rho^2 + 2\rho\sigma + \sigma^2 \geqslant \frac{1}{4}\kappa^2,$$

and

$$(4\rho^2 + 4\sigma^2 - \kappa^2)^2 \geqslant 64\rho^2\sigma^2,$$

so by (29) the second factor of (28) vanishes if I is on ON, but is positive for I elsewhere. Also the first factor α^2 vanishes only if $\sigma = 0$. That would place I at N, the limiting equilateral case already considered.

Hence the roots are real only if the third factor of (28) is negative. Expressed in terms of ρ, σ, κ, this gives

$$27\rho^4 - 40\rho^2\sigma^2 + 16\sigma^4 - 8\sigma^2\kappa^2 \leqslant 0.$$

This inequality determines a region of the plane whose boundary has the bipolar equation

$$27\rho^4 - 40\rho^2\sigma^2 + 16\sigma^4 - 8\sigma^2\kappa^2 = 0.$$

This is precisely the form taken by (23) when $c = -1$. That is the extremal quartic $Q(\pi)$, which leads to the following:

DEFINITION 4. The acentric lacuna is the region inside the extremal quartic $Q(\pi)$.

Thus all positions of I inside the acentric lacuna except those on the Euler line lead to cubics (23) with complex roots, and are therefore impossible locations for tritangent centers of a real triangle. If I is on the Euler line and inside the acentric lacuna, then its cartesian coordinates satisfy $-1 < x < 1/3, y = 0$. For $y = 0$ the equation (23) becomes

$$(2cx + x - 1)^2(2cx^2 - x^2 - 2x + 1) = 0.$$

Its roots in c are therefore $(x^2 + 2x - 1)/2x^2$, $(1 - x)/2x$, the latter repeated. These all lie outside the permissible range $-1 \leqslant c \leqslant 1$ so no position of I inside the acentric lacuna is possible.

Positions outside the acentric lacuna and not on the critical circle or the extremal quartic are either inside the critical circle or outside both critical circle and extremal quartic. The center of the critical circle is $(2/3, 0)$ and the corresponding roots of (26) are $\frac{1}{4}, \frac{1}{4}, \frac{7}{8}$; all within the permissible range for cosines of real angles. Since these roots are continuous functions of the position of I, they cannot exceed 1 unless I moves across $Q(0)$, the critical circle, and they cannot go below -1 unless I crosses $Q(\pi)$. Hence all I within the critical circle are permissible as incenters of real triangles (provided N is regarded as a limiting case.)

Similarly if I is at $(2, 0)$, outside both critical circle and acentric lacuna, then the roots of (26) are $-\frac{1}{4}, -\frac{1}{4}, \frac{7}{8}$, also within permissible range, and a similar continuity argument applies. We thus have:

THEOREM 4. *No tritangent center can lie within the acentric lacuna, but all points outside the lacuna and outside the critical circle can be excenters of real triangles. All points inside the critical*

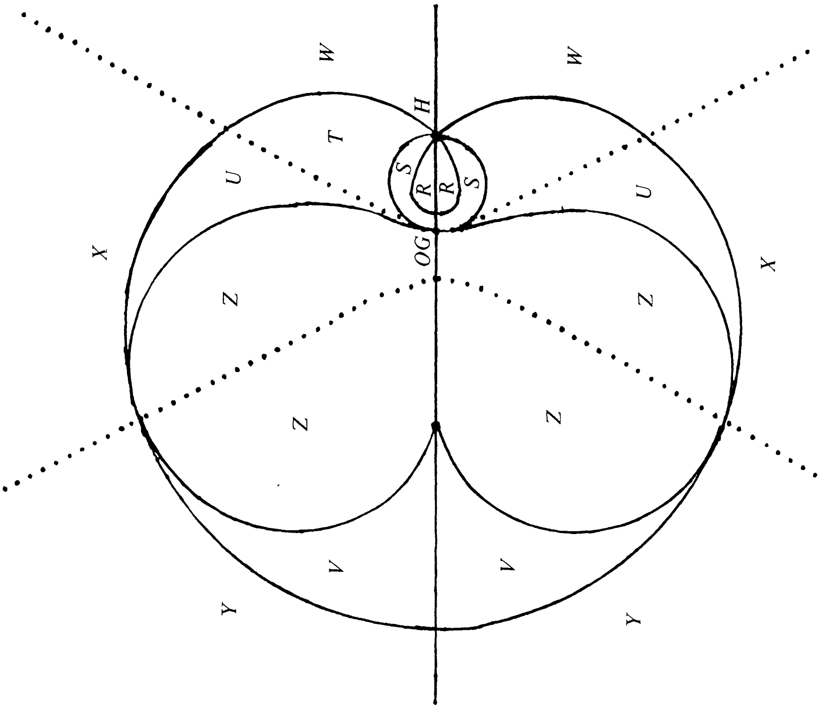

FIG. 3.

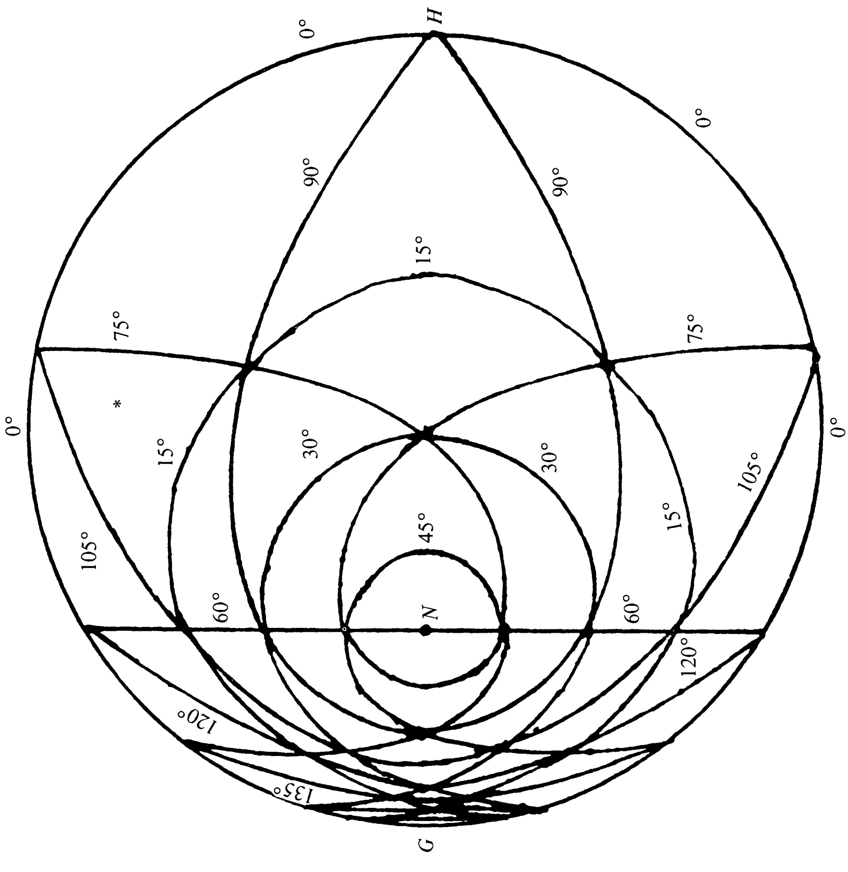

FIG. 2.

circle except the nine-point center can be incenters of real, finite triangles. Points on the borders between these regions can be regarded as excenters of degenerate triangles, and the nine-point center can be regarded as corresponding to an infinite equilateral triangle.

9. Tritangent Centers and Their Regions. Fig. 2 shows the interior of the critical circle partitioned into regions by the curves $Q(\theta)$ for θ at intervals of $15°$. If the position of a tritangent center is given in one of these regions, then it must be an incenter, and the angles of the corresponding triangle must lie in ranges determined by the $Q(\theta)$ surrounding that region. For example, the point marked * lies inside a curvilinear triangle. One side is part of $Q(15°)$ and the opposite vertex is on $Q(0°)$; hence one angle of the triangle with incenter * is in the range $0°$ to $15°$. Similarly the other angles of the triangle are in the ranges $60°$ to $75°$, and $90°$ to $105°$. A finer network of $Q(\theta)$ would narrow these ranges.

For tritangent centers outside the critical circle, the nature of the center can be determined from the region in which it lies. Fig. 3 shows the Euler line, the critical circle, the triage-hyperbola, the extremal quartic, and the limaçon $Q(\frac{1}{2}\pi)$. These suffice to partition the plane into regions as shown in Table 1.

TABLE 1.

If I is in region:	then it is:	and the triangle is:
R	the incenter	acute-angled
S	the incenter	obtuse-angled
T	the excenter opposite the least angle	obtuse-angled
U	the excenter opposite the middle angle	obtuse-angled
V	the excenter opposite the greatest angle	obtuse-angled
W	the excenter opposite the least angle	acute-angled
X	the excenter opposite the middle angle	acute-angled
Y	the excenter opposite the greatest angle	acute-angled
Z	not a tritangent center	

10. Types of Tritangent Centers on Individual $Q(\theta)$. A tritangent center I on a particular quartic $Q(\theta)$ can be of the following three types:

(a) incenter of a triangle with an angle θ;
(b) excenter opposite an angle θ of the triangle;
(c) excenter of a triangle with an angle $\pi - \theta$, but not opposite that angle.

If I moves along $Q(\theta)$, then changes between types occur as I crosses the critical circle or touches the extremal quartic. The pattern of the changes depends on the value of θ, so it is best explained by an example. Fig. 4 shows $Q(80°)$, with various special points on it. The points and corresponding triangles are as shown in Table 2; the angles $\theta = 80°$ and $\pi - \theta = 100°$ concerned are underlined.

Changes of type of center follow the same pattern for $60° < \theta < 120°$. For other ranges of θ the pattern can be sorted out in similar fashion.

11. Remarks and Recent History. Restrictions on the location of the incenter have been noted previously in various forms. Bottema [**6**] attributes $I_0H\sqrt{2} \leqslant OH$ to S. G. Guba; Radford [**2**] shows that OI_0H is obtuse; Bankoff [**2**], [**3**] that I_0 lies in BOH if $A < B < C$; Blundon [**4**] that the projection of I_0 on the Euler line lies between G and H, and that GI_0H is obtuse.

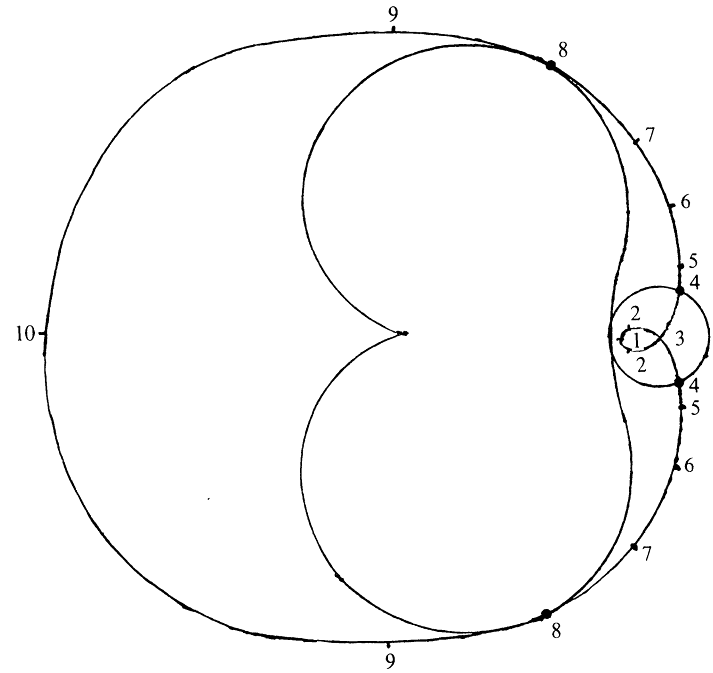

FIG. 4.

TABLE 2.

Point and location	Angles of the triangle	Type	Opposite angle
1. Euler line.	50°, 50°, 80°	(a)	—
2. $Q(60°)$, $x = \frac{1}{2}$.	40°, 60°, 80°	(a)	—
3. node.	20°, 80°, 80°	(a)	—
4. critical circle.	0°, 100°, 80°	(a) & (c)	0°
5. bitangent contact.	8.4°, 100°, 71.6°	(c)	8.4°
6. triage-hyperbola.	40°, 100°, 40°	(c)	40°
7. $Q(60°)$, $x = \frac{1}{2}$.	60°, 100°, 20°	(c)	60°
8. contact $Q(180°)$	80°, 100°, 0°	(c) & (b)	80°
9. triage-hyperbola.	80°, 80°, 20°	(b)	80°
10. Euler line.	80°, 50°, 50°	(b)	80°

The last three writers all commented on confusion caused by a problem in Hobson's *Trigonometry* [**12**]. In the present notation the problem is:

If OI_0H is an equilateral triangle, show that

$$\cos A + \cos B + \cos C = 3/2.$$

Both premise and conclusion imply that ABC is equilateral and hence that OI_0H degenerates to a point. It seems unlikely that Hobson had this in mind.

I have been unable to find previous mention of the rôle of the critical circle in separating incenter from excenters; still less any hint of the existence of an acentric region.

It follows readily from Theorem 2 that no general ruler-and-compass reconstruction of a triangle from Euler line and a tritangent center is possible [**11**]. The present investigation stems from one of a number of triangle reconstruction problems posed by Wernick [**18**]. With respect to the triangle, it is the most symmetric of Wernick's problems; that is probably why it leads so unexpectedly to the critical circle and the acentric lacuna.

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MISCELLANEA

127.

The mathematics that I learned... became a deeper part of my nature, as resolute consistency and something like mental courage. From a possibly very small area, which is not to be doubted, you keep on going in one and the same direction, never asking yourself where you might end up, refusing to look right or left, as though heading towards some goal without knowing which, and so long as you make no false step and maintain the connection of the steps, nothing will happen to you, you progress into the unknown—the only way to conquer the unknown *gradually*.

—Elias Canetti, *The Tongue Set Free*,
The Seabury Press, New York, 1979,
pp. 236–237

ANSWER TO PHOTO ON PAGE 283

Béla Szökefalvi-Nagy and Ciprian Foiaş, usually known as Nagy-Foiaş.