Vektori
(najvažniji zadaci za ponavljanje)

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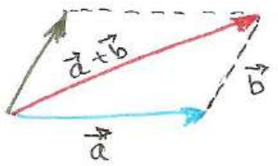
VEKTORI

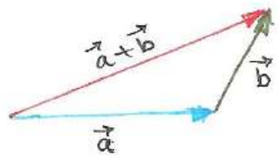
-formule

Zbroj vektora

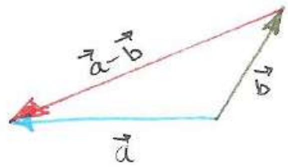
→ pravilo paralelograma



→ pravilo trokuta



oduzimanje vektora



$\boxed{\vec{a} = k \cdot \vec{b}}$ uvjet kolinearnosti

$$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$$

$$|\vec{AB}| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$$

$$\boxed{\begin{array}{l} \vec{a} = a_x \vec{i} + a_y \vec{j} \\ |\vec{a}| = \sqrt{a_x^2 + a_y^2} \end{array}}$$

jedinični vektor

$$\boxed{\vec{e} = \frac{\vec{a}}{|\vec{a}|}}$$

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi}$$

$$\left.\begin{array}{l} \vec{a} = a_x \vec{i} + a_y \vec{j} \\ \vec{b} = b_x \vec{i} + b_y \vec{j} \end{array}\right\} \Rightarrow \boxed{\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y}$$

$$\vec{i} \cdot \vec{i} = |\vec{i}| \cdot |\vec{i}| \cdot \cos(0^\circ) = 1$$

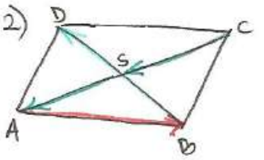
$$\vec{j} \cdot \vec{j} = |\vec{j}| \cdot |\vec{j}| \cdot \cos(0^\circ) = 1$$

$$\vec{i} \cdot \vec{j} = |\vec{i}| \cdot |\vec{j}| \cdot \cos(90^\circ) = 0$$

str. 15.

(2.)

2)



$\overrightarrow{AB}+\overrightarrow{CS}+\overrightarrow{BD}=?$

$\overrightarrow{AB}+\overrightarrow{CS}+\overrightarrow{BD}$

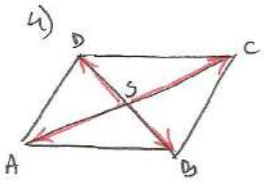
$=\overrightarrow{AB}+\overrightarrow{SA}+\overrightarrow{BD}$

$=\overrightarrow{SA}+\overrightarrow{AB}+\overrightarrow{BD}$

$=\overrightarrow{SB}+\overrightarrow{BD}$

$=\overrightarrow{SD}$

4)



$\overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC}+\overrightarrow{SD}=?$

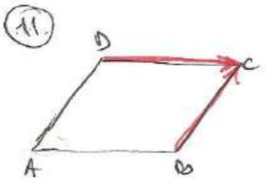
$\overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{SC}+\overrightarrow{SD}$

$=\overrightarrow{SA}+\overrightarrow{SB}+\overrightarrow{AS}+\overrightarrow{BS}$

$0 \qquad 0$

$=\vec{0}$

(11.)

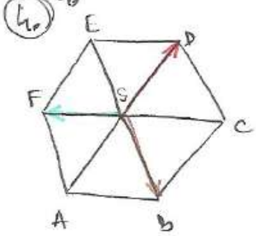


1) $\overrightarrow{BC}-\overrightarrow{DC}$

$=\overrightarrow{BC}+\overrightarrow{CD}$

$=\overrightarrow{BD}$

(4.) 4)



pravilni šesterokut

$\overrightarrow{SB}+\overrightarrow{SD}+\overrightarrow{SF}$

$=\overrightarrow{SB}+\overrightarrow{SD}+\overrightarrow{CS}$

$=\overrightarrow{SB}+\overrightarrow{CS}+\overrightarrow{SD}$

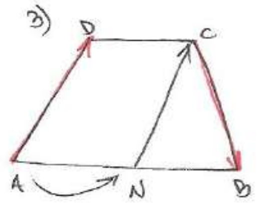
$=\overrightarrow{SB}+\overrightarrow{CD}$

$=\overrightarrow{SB}+\overrightarrow{BS}$

$=\vec{0}$

(5.)

3)



$\overrightarrow{AD}+\overrightarrow{CB}$

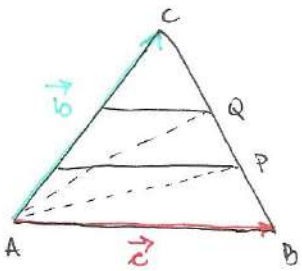
$=\overrightarrow{NC}+\overrightarrow{CB}$

$=\overrightarrow{NB}$

str. 25.

(4.)

Stranica $\overline{BC}$ trokuta ABC točkama P i Q podijeljena je na tri jednaka dijela. Izrazi vektore $\overrightarrow{AP}$ i $\overrightarrow{AQ}$ kao linearnu kombinaciju vektora $\overrightarrow{AB}=\vec{c}$ i $\overrightarrow{AC}=\vec{b}$.



$\overrightarrow{BC}=-\vec{c}+\vec{b}=\vec{b}-\vec{c}$

$\overrightarrow{AP}=\vec{c}+\frac{1}{3}\overrightarrow{BC}$

$=\vec{c}+\frac{1}{3}(\vec{b}-\vec{c})$

$=\vec{c}+\frac{1}{3}\vec{b}-\frac{1}{3}\vec{c}$

$=\frac{1}{3}\vec{b}+\frac{2}{3}\vec{c}$

$\overrightarrow{AQ}=\overrightarrow{AB}+\frac{2}{3}\overrightarrow{BC}$

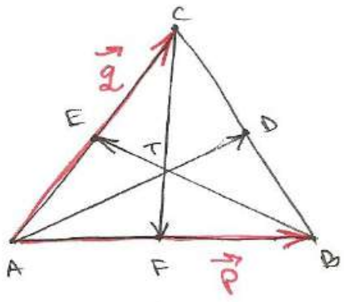
$=\vec{c}+\frac{2}{3}(\vec{b}-\vec{c})$

$=\vec{c}+\frac{2}{3}\vec{b}-\frac{2}{3}\vec{c}$

$=\frac{2}{3}\vec{b}+\frac{1}{3}\vec{c}$

5. Dužine $\overline{AD}$, $\overline{BE}$ i $\overline{CF}$ težišnice su trokuta ABC.
Izrazi vektore $\vec{AD}$, $\vec{BE}$ i $\vec{CF}$ kao linearnu kombinaciju vektora $\vec{AB}=\vec{p}$ i $\vec{AC}=\vec{q}$.

- ono što je važno znati o težišnici za ovaj zadatak je da je to dužina koja spaja vrh s polovištem nasuprotne stranice i da se sve tri težišnice trokuta sijeku u jednoj točki
- ostatak o težišnici
  - dijeli trokut na 2 dijela jednake površine
  - težište T dijeli svaku od težišnica u omjeru 1:2



$$\vec{BC} = \vec{BA} + \vec{AC} = -\vec{p} + \vec{q} = \vec{q} - \vec{p}$$

$$\begin{aligned}\vec{AD} &= \vec{AB} + \tfrac{1}{2}\vec{BC} \\ &= \vec{AB} + \tfrac{1}{2}(\vec{q} - \vec{p}) \\ &= \vec{p} + \tfrac{1}{2}\vec{q} - \tfrac{1}{2}\vec{p} \\ &= \underline{\underline{\tfrac{1}{2}\vec{p} + \tfrac{1}{2}\vec{q}}}\end{aligned}$$

$$\vec{BE} = \vec{BA} + \vec{AE} = \underline{\underline{-\vec{p} + \tfrac{1}{2}\vec{q}}}$$

$$\vec{CF} = \vec{CA} + \vec{AF} = \underline{-\vec{q} + \tfrac{1}{2}\vec{p}}$$

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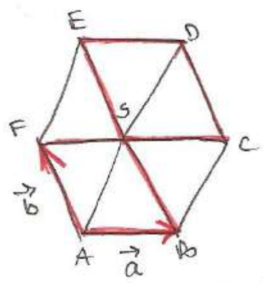
6. pravilni šesterokut. Izrazi vektore $\vec{BC}$ i $\vec{BD}$ kao linearnu kombinaciju vektora $\vec{AB}$ i $\vec{AF}$.



$$\begin{aligned}\vec{BC} &= \vec{BS} + \vec{SC} \\ &= \vec{b} + \vec{a}\end{aligned}$$

$$\begin{aligned}\vec{BD} &= \vec{BS} + \vec{SC} + \vec{CD} \\ &= \vec{b} + \vec{a} + \vec{b} \\ &= \vec{a} + 2\vec{b}\end{aligned}$$

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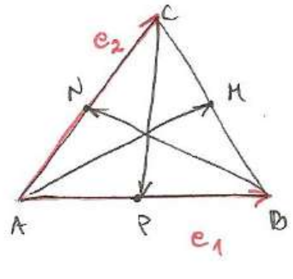
8. M, N i P su polovišta stranica $\overline{BC}$, $\overline{AC}$ i $\overline{AB}$ trokuta ABC. Prikaži vektore $\vec{AM}$, $\vec{BN}$ i $\vec{CP}$ kao linearne kombinacije vektora $\vec{AB}=\vec{e}_1$ i $\vec{AC}=\vec{e}_2$.



$$\vec{AM} = \vec{AN} + \vec{NM} = \tfrac{1}{2}\vec{e}_2 + \tfrac{1}{2}\vec{e}_1$$

$$\vec{BN} = \vec{BA} + \vec{AN} = -\vec{e}_1 + \tfrac{1}{2}\vec{e}_2$$

$$\vec{CP} = \vec{CA} + \vec{AP} = -\vec{e}_2 + \tfrac{1}{2}\vec{e}_1$$

(13.) Ako su $\vec{m}$ i $\vec{n}$ nekolinearni vektori, odredi realni broj x tako da vektori $\vec{a}$ i $\vec{b}$, $\vec{a}=(x-1)\vec{m}+\vec{n}$ i $\vec{b}=3\vec{m}+(x+1)\vec{n}$ budu kolinearni.

$\vec{a}=(x-1)\vec{m}+\vec{n}$

$\vec{b}=3\vec{m}+(x+1)\vec{n}$

$x=?$ tako da su $\vec{a}$ i $\vec{b}$ kolinearni

$\vec{a}=k\cdot\vec{b}$ } uvjet kolinearnosti

$(x-1)\vec{m}+\vec{n}=k\cdot(3\vec{m}+(x+1)\vec{n})$

$(x-1)\vec{m}+\vec{n}=3k\vec{m}+k(x+1)\cdot\vec{n}$

$$\begin{cases} x-1=3k \longrightarrow k=\dfrac{x-1}{3} \\ 1=k(x+1) \end{cases}$$

$1=\frac{x-1}{3}(x+1)$

$\frac{x-1}{3}(x+1)=1 \quad /\cdot 3$

$(x-1)(x+1)=3$

$x^2-1=3$

$x^2=4$

$x_1=2$	$x_2=-2$
$k_1=\frac{x_1-1}{3}$	$k_2=\frac{x_2-1}{3}$
$k_1=\frac{1}{3}$	$k_2=-1$
$\vec{a}=\frac{1}{3}\vec{b}$	$\vec{a}=-\vec{b}$
$3\vec{a}=\vec{b}$	

Za $x=2$ je $\vec{b}=3\vec{a}$, a za $x=-2$ je $\vec{a}=-\vec{b}$.

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str. 29. – Vektori u Kartezijevu koordinatnom sustavu

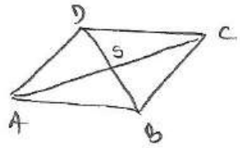
(11.)

$B(1,-2)$

$C(3,2)$

$S\left(-\frac{1}{2},\frac{3}{2}\right)$

$A, D=?$



$\overrightarrow{AS}=\overrightarrow{SC}$

$\left(-\frac{1}{2}-x_A\right)\vec{i}+\left(\frac{3}{2}-y_A\right)\vec{j}=\left(3+\frac{1}{2}\right)\vec{i}+\left(2-\frac{3}{2}\right)\vec{j}$

$\left(-\frac{1}{2}-x_A\right)\vec{i}+\left(\frac{3}{2}-y_A\right)\vec{j}=\frac{7}{2}\vec{i}+\frac{1}{2}\vec{j}$

$$\begin{cases} -\frac{1}{2}-x_A=\frac{7}{2} \\ \frac{3}{2}-y_A=\frac{1}{2} \end{cases}$$

$x_A=-4$

$y_A=1$

$\boxed{A(-4,1)}$

$\overrightarrow{BS}=\overrightarrow{SD}$

$\left(-\frac{1}{2}-1\right)\vec{i}+\left(\frac{3}{2}+2\right)\vec{j}=\left(x_D+\frac{1}{2}\right)\vec{i}+\left(y_D-\frac{3}{2}\right)\vec{j}$

$-\frac{3}{2}\vec{i}+\frac{7}{2}\vec{j}=\left(x_D+\frac{1}{2}\right)\vec{i}+\left(y_D-\frac{3}{2}\right)\vec{j}$

$$\begin{cases} -\frac{3}{2}=x_D+\frac{1}{2} \\ \frac{7}{2}=y_D-\frac{3}{2} \end{cases}$$

$x_D=-2$

$y_D=5$

$\boxed{D(-2,5)}$

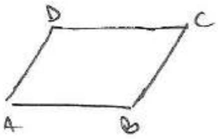
13.

$A(3,2)$

$B(1,-2)$

$D(5,1)$

$\overline{AC} = ?$



$\vec{AB} = \vec{DC}$

$(1-3)\vec{i} + (-2-2)\vec{j} = (x_C-5)\vec{i} + (y_C-1)\vec{j}$

$-2\vec{i} - 4\vec{j} = (x_D-5)\vec{i} + (y_C-1)$

$\begin{cases} -2 = x_C - 5 \\ -4 = y_C - 1 \end{cases}$

$x_C = 3$

$y_C = -3$

$C(3,-3)$

$|AC| = \sqrt{(x_C-x_A)^2 + (y_C-y_A)^2}$

$|AC| = \sqrt{(3-3)^2 + (-3-2)^2}$

$|AC| = 5$

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17. Odredi jedinični vektor koji ima isti smjer, ali suprotnu orijentaciju od vektora $\vec{AB}$, $A(-4,9)$, $B(-2,5)$.

$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$

$= (-2+4)\vec{i} + (5-9)\vec{j}$

$= 2\vec{i} - 4\vec{j}$

$|\vec{AB}| = \sqrt{x^2+y^2}$

$= \sqrt{2^2 + (-4)^2}$

$= 2\sqrt{5}$

općenito $\vec{e} = \dfrac{\vec{AB}}{|AB|}$

$\vec{e} = -\dfrac{\vec{AB}}{|AB|} = -\dfrac{2\vec{i} - 4\vec{j}}{2\sqrt{5}} = -\dfrac{\cancel{2}(\vec{i} - 2\vec{j})}{\cancel{2}\sqrt{5}} = \dfrac{-\vec{i} + 2\vec{j}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{-\sqrt{5}\vec{i} + 2\sqrt{5}\vec{j}}{5} = -\dfrac{\sqrt{5}}{5}\vec{i} + \dfrac{2\sqrt{5}}{5}\vec{j}$

suprotna orijentacija!

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18. Odredi vektor $\vec{v}$ kolinearan s vektorom $\vec{AB}$, gdje je $A(2,-1)$, $B(-1,3)$ ako je $|\vec{v}| = 20$.

$A(2,-1)$

$B(-1,3)$

$|\vec{v}| = 20$

$\vec{v} = ?$

$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$

$= -3\vec{i} + 4\vec{j}$

$|\vec{AB}| = \sqrt{AB_x^2 + AB_y^2}$

$= 5$

$\vec{e} = \dfrac{\vec{AB}}{|\vec{AB}|} = -\dfrac{3}{5}\vec{i} + \dfrac{4}{5}\vec{j}$

$\vec{e} = \dfrac{\vec{v}}{|\vec{v}|}$

ne znamo orijentaciju $\leftarrow$ $\vec{v} = \pm |\vec{v}| \cdot \vec{e}$

$\vec{v} = \pm 20 \cdot \left(-\dfrac{3}{5}\vec{i} + \dfrac{4}{5}\vec{j}\right)$

$\boxed{\begin{aligned} \vec{v}_1 &= -12\vec{i} + 16\vec{j} \\ \vec{v}_2 &= 12\vec{i} - 16\vec{j} \end{aligned}}$

(23.) Odredi ordinatu y točke $B(0,y)$ tako da ta točka pripada pravcu AC,
$A(-1,7)$, $C(2,1)$

$A(-1,7)$
$C(2,1)$
$B(0,y_B)$
$y_B=?$

$B\in \overrightarrow{AC} \Rightarrow \overrightarrow{AB}=k\cdot\overrightarrow{AC}$

$(x_B-x_A)\vec{i}+(y_B-y_A)\vec{j}=k\left((x_C-x_A)\vec{i}+(y_C-y_A)\vec{j}\right)$

$(0+1)\vec{i}+(y_B-7)\vec{j}=k\cdot((2+1)\vec{i}+(1-7)\vec{j})$

$\vec{i}+(y_B-7)\vec{j}=3k\vec{i}-6k\vec{j}$

$1=3k \longrightarrow k=\frac{1}{3}$

$y_B-7=-6k$

$y_B-7=-6\cdot\frac{1}{3}$

$y_B=-2+7$

$y_B=5$

$\boxed{B(0,5)}$

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str. 36 i 37 → Skalarni umnožak

(6.) $\vec{a}=2\vec{i}-\vec{j}$

$\vec{b}=3\vec{i}+2\vec{j}$

$\vec{a}\cdot\vec{c}=7$

$\vec{b}\cdot\vec{c}=7$

$\vec{c}=?$

$\vec{c}=c_x\vec{i}+c_y\vec{j}$

$\vec{a}\cdot\vec{c}=7$

$(2\vec{i}-\vec{j})\cdot(c_x\vec{i}+c_y\vec{j})=7$

$\vec{i}\cdot\vec{j}=0$ (jer su pod 90°)

$2c_x\vec{i}^2+\underbrace{2c_y\vec{i}\vec{j}}_{0}-\underbrace{c_x\vec{i}\vec{j}}_{0}-c_y\vec{j}^2=7$

$2c_x\underbrace{\vec{i}^2}_{1}-c_y\underbrace{\vec{j}^2}_{1}=7$

$\boxed{2c_x-c_y=7}$

$\vec{b}\cdot\vec{c}=7$

$(3\vec{i}+2\vec{j})\cdot(c_x\vec{i}+c_y\vec{j})=7$

$3c_x\underbrace{\vec{i}^2}_{1}+3c_y\underbrace{\vec{i}\vec{j}}_{0}+2c_x\underbrace{\vec{i}\vec{j}}_{0}+2c_y\underbrace{\vec{j}^2}_{1}=7$

$\boxed{3c_x+2c_y=7}$

$\begin{cases}2c_x-c_y=7 \;/\cdot 2\\ 3c_x+2c_y=7\end{cases}$

$4c_x-\cancel{2c_y}=14$

$3c_x+\cancel{2c_y}=7$ / +

$7c_x=21$

$\boxed{c_x=3}$

$2c_x-c_y=7$

$6-c_y=7$

$-c_y=1$

$\boxed{c_y=-1}$

$\boxed{\vec{c}=3\vec{i}-\vec{j}}$

9.

$\vec{p} = 3\vec{i} - 2\vec{j}$

$\vec{q} = -\vec{i} + 4\vec{j}$

$\sphericalangle(\vec{p}+\vec{q}, \vec{p}-\vec{q}) = ?$

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi$

$\cos\varphi = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

$\vec{a} = \vec{p}+\vec{q} = 3\vec{i} - 2\vec{j} - \vec{i} + 4\vec{j}$

$= 2\vec{i} + 2\vec{j}$

$\vec{b} = \vec{p} - \vec{q} = 3\vec{i} - 2\vec{j} + \vec{i} - 4\vec{j}$

$= 4\vec{i} - 6\vec{j}$

$\varphi = \sphericalangle(\vec{a}, \vec{b}) = ?$

$|\vec{a}| = \sqrt{a^2_x + a^2_y} = \sqrt{2^2+2^2} = 2\sqrt{2}$

$|\vec{b}| = \sqrt{b^2_x + b^2_y} = \sqrt{4^2 + (-6)^2} = 2\sqrt{13}$

$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y$

$\vec{a}\cdot\vec{b} = 2\cdot 4 + 2\cdot(-6)$

$\vec{a}\cdot\vec{b} = -4$

$\cos\varphi = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$

$\cos\varphi = -\dfrac{1}{\sqrt{26}}$

$\boxed{\varphi = 101^\circ 18'}$

16.

A (1,4)

B (3,3)

C (−5,3)

D (1,5)

$\varphi = \sphericalangle(\vec{AB}, \vec{CD}) = ?$

$\cos\varphi = \dfrac{\vec{AB}\cdot\vec{CD}}{|\vec{AB}|\cdot|\vec{CD}|}$

$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$

$= 2\vec{i} - \vec{j}$

$\vec{CD} = (x_D - x_C)\vec{i} + (y_D - y_C)\vec{j}$

$= 6\vec{i} + 2\vec{j}$

$|\vec{AB}| = \sqrt{AB^2_x + AB^2_y}$

$= \sqrt{2^2 + (-1)^2}$

$= \sqrt{5}$

$|\vec{CD}| = \sqrt{CD^2_x + CD^2_y}$

$= \sqrt{6^2 + 2^2}$

$= 2\sqrt{10}$

$\cos\varphi = \dfrac{\vec{AB}\cdot\vec{CD}}{|\vec{AB}|\cdot|\vec{CD}|}$

$\cos\varphi = \dfrac{(2\vec{i} - \vec{j})(6\vec{i} + 2\vec{j})}{\sqrt{5}\cdot 2\sqrt{5}}$

$\cos\varphi = \dfrac{12\vec{i}^2 + 4\vec{i}\vec{j} - 6\vec{i}\vec{j} - 2\vec{j}^2}{2\sqrt{50}}$

$\cos\varphi = \dfrac{12-2}{10\sqrt{2}}$

$\cos\varphi = \dfrac{1}{\sqrt{2}}$ $\boxed{\varphi = 45^\circ}$

27. Odredi jedinični vektor okomit na vektor $\vec{AB}$ ako je $A(-2,3)$, $B(-4,2)$.

$A(-2,3)$

$B(-4,2)$

$\vec{e} \perp \vec{AB} \Rightarrow \vec{e} \cdot \vec{AB} = 0$

$\vec{e} = ?$

$\vec{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j}$

$\vec{AB} = -2\vec{i} - \vec{j}$

$\vec{e} = e_x\vec{i} + e_y\vec{j}$

$\vec{AB} \cdot \vec{e} = 0$

$-2e_x - e_y = 0$

$e_y = -2e_x$

$|\vec{e}| = \sqrt{e_x^2 + e_y^2}$

$1 = \sqrt{e_x^2 + (-2e_x)^2} \quad /^2$

$1 = e_x^2 + 4e_x^2$

$1 = 5e_x^2$

$e_x^2 = \frac{1}{5} \quad /\sqrt{\ }$

$e_{x1} = \frac{1}{\sqrt{5}}$

$e_{x2} = -\frac{1}{\sqrt{5}}$

$e_{y1} = -\frac{2}{\sqrt{5}}$

$e_{y2} = \frac{2}{\sqrt{5}}$

$$\boxed{\begin{aligned} \vec{e}_1 &= \frac{1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j} \\ \vec{e}_2 &= -\frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j} \end{aligned}}$$

28.

$|\vec{a}| = 11$

$|\vec{b}| = 23$

$|\vec{a} - \vec{b}| = 30$

$|\vec{a} + \vec{b}| = ?$

$|\vec{a} + \vec{b}| = \sqrt{(\vec{a} + \vec{b})^2} \quad /^2$

$|\vec{a} + \vec{b}| = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2$

$|\vec{a} - \vec{b}| = \sqrt{(\vec{a} - \vec{b})^2} \quad /^2$

$|\vec{a} - \vec{b}|^2 = \vec{a}^2 - 2\vec{a}\vec{b} + \vec{b}^2$

$|\vec{a} + \vec{b}|^2 = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2$

$|\vec{a} - \vec{b}|^2 = \vec{a}^2 - 2\vec{a}\vec{b} + \vec{b}^2 \quad / +$

$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2\vec{a}^2 + 2\vec{b}^2$

$|\vec{a} + \vec{b}|^2 + 30^2 = 2 \cdot 11^2 + 2 \cdot 23^2$

$|\vec{a} + \vec{b}|^2 = 400 \quad /\sqrt{\ }$

$|\vec{a} + \vec{b}| = 20$