Average Speed for Gravity Tunnel Simple Harmonic Motion

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Introduction

For simple harmonic motion (SHM), and the Gravity Tunnel applet in particular, it is of interest to see how the speed of the object relates to the length of the object's travel during a cycle. The motion takes the same time, T, the period of the SHM, to traverse a long path or a shorter path. Put another way, the period T is independent of the initial displacement, which sets the path length of the motion. The only way this is possible is that *the speed of the object must vary directly with the path length*. The objective of this study is to derive that relation.

We do not use the *velocity*, since it is a vector; this is one-dimensional motion and so the velocity vector will have an alternating positive or negative sign. The average velocity will thus be zero, which is not especially useful. So we will use the absolute value of the velocity, the *speed*. The velocity, speed, and indeed even the acceleration, are not constant during the SHM cycle; see the accompanying PDF which derives the SHM relations for the Gravity Tunnel. Thus the usual kinematic formulas will not be applicable, since they are based on the assumption of a constant acceleration. For a varying quantity- here, the speed of the object undergoing SHM- we can use the average, or alternatively the maximum, to get a single value to represent that changing quantity.

Development

Denote the initial displacement of the object from its rest (equilibrium) position as x_0. This displacement, of course, is what initiates the motion of the object. The larger this x_0, the longer the path of the motion, which goes between $+x_0$ and $-x_0$, with the equilibrium position at zero. It is shown in the companion PDF that the motion in the tunnel is given by

$$x(t) = x_0 * Cos(\omega\ t) \tag{1}$$

where x is measured along the length of the tunnel, no matter how it is oriented on the spherical planet. The parameter ω is determined by the properties of the planet (its mass and radius) and so is a constant. Then the velocity of the object will be

$$v(t) = dx/dt = -\ x_0 * \omega * Sin(\omega\ t) \tag{2}$$

and the speed is the absolute value of this; see the figure below for a plot of the speed vs. time.

The simplest measure of the changing speed is to recognize in Eq(2) that the maximum value of the sine function is unity, so then the *maximum speed will be* $x_0 * \omega$. This maximum speed depends on both the initial displacement and the properties of the planet. The larger the initial displacement, the longer the path (length $2*x_0$), but at the same time the object moves faster.

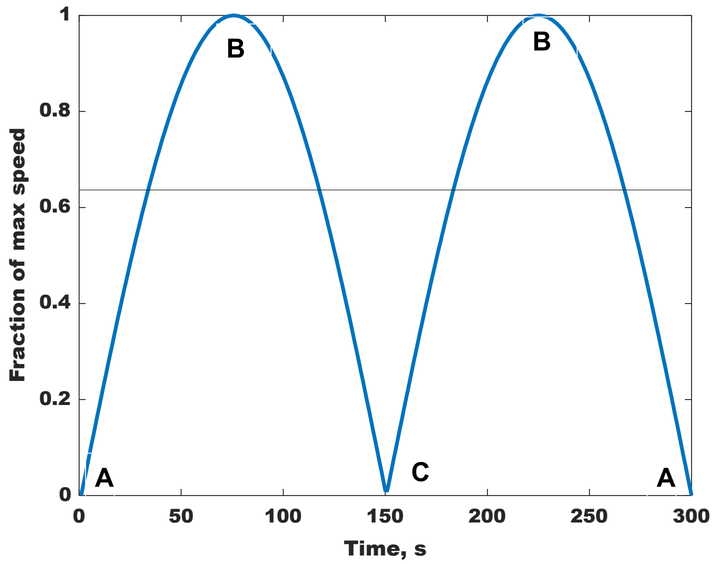
Average Speed

We find the average speed over one full cycle, of time T, using a result from calculus (see, e.g., Thomas/Finney, *Calculus*, 9th Ed., p. 328). Note that in the integral below we do not mix w and T, since they are related; the integration is done with one or the other. In this instance we change ω to express it in terms of T:

$$\text{average speed} = (1/T) * \text{Integral of } Abs[\ -x_0 * 2\pi / T * Sin(\ 2\pi * t / T\ )\ ],\ \text{for } t \text{ from } 0 \text{ to } T\ =\ 4*Abs(x_0) / T \tag{3}$$

So this measure of the changing speed, its average, like the maximum speed also varies directly with the initial displacement x_0. We will keep the absolute value of x_0 since it could be "above" (positive) or "below" (negative) the equilibrium position. And again, a larger x_0 means a longer path length ($2\ x_0$), but the object will simultaneously have a higher speed.

We can also just use common sense to find the average speed; see the figure. Take the starting position of the object, at its initial displacement x_0, to be A; the equilibrium (rest) position is B; the farthest position from A is C. The motion then consists of (1) A to B, length x_0; (2) B to C, x_0; (3) C back to B, x_0; (4) B back to A, x_0. So there were 4 x_0 distances covered, in time T. In spite of what we said above about kinematics formulas, there is one simple formula that always works: the average speed is just the total distance divided by the total time. Which is of course $4x_0/T$, exactly as we found with calculus, above. Note in the figure that the speed is zero at the endpoints of the motion (A and C), and maximum as the object crosses the zero position (B).



Absolute value of normalized speed vs. time for one cycle. Thin horizontal line is the average, 2/π.

Energy

Displacing the object from its equilibrium position requires applying a force, one that opposes the "restoring force" which keeps the object at its rest position, over some distance. This means that work, in the physics sense of the word, is done, and so potential energy is added to the system. When the object is released and the motion starts, this potential energy transfers to kinetic energy, and we can write the conservation of energy relation

$$½ * m * v(t)^2 \quad + \quad ½ * k * m * x(t)^2 \quad = \quad ½ * k * m * x_0^2 \tag{4}$$

which says that the sum of the kinetic and potential energies at any time equals the energy input to the system at time zero. Since the object is *released* rather than *thrown*, there is no initial kinetic energy, so this initial energy is entirely potential. The potential energy for the gravity tunnel is based on a spring-mass analogy. We integrate the force F that is *applied*, to move the object (a positive force, opposite in direction to the negative gravitational force of attraction toward the planet center) along the tunnel over a displacement x,

$$\text{Work done} = \text{PE added} = \text{Integrate } F_{applied} = + k*m*x \text{ over some range of } x \quad = \quad ½ * k * m * x^2,$$

which was used in Eq(4); see the companion PDF for a derivation of the force F and definition of k (it is a combination of planet-specific parameters). Simplifying and re-arranging Eq(4),

$$v(t)^2 \; = \; k * (\, x_0^2 \; - \; x(t)^2 \,) \; = \; k * [\, x_0^2 \; - \; x_0^2 * Cos^2(\omega\, t) \,] \; = \; k * x_0^2 * [\, 1 - Cos^2(\omega\, t) \,] \; = \; k * x_0^2 * Sin^2(\omega\, t)$$

and then

$$v(t) = \sqrt{k} * x_0 * Sin(\omega\, t) \quad = \quad \pm\, x_0 * w * Sin(\, \omega\, t\,)$$

since $\sqrt{k} = \omega$. The velocity can have either sign (due to the square root), so again we use the absolute value to obtain the speed. Then inspection will show that if we set up an integral for the average speed using this result, we will have exactly the same setup as in Eq(3), and of course obtain the same result.

Conclusion

It has been shown that the average speed of an object moving in a gravity tunnel varies directly with the size of the initial displacement of the object. The maximum speed is also directly proportional to that initial displacement. These results explain how this SHM takes the same time, the period T, to traverse either a long or a short path in the tunnel. If the path is longer, the speed is higher, and if the path is shorter, the speed will be proportionally slower.

With the appropriate adjustments these results, particularly Eq(3), should be applicable to other examples of SHM.