**TRANFORMATIONS**

**Translation**  
\[ (x', y') = (x + a, y + b) \]

**Reflection**  
\[ (x', y') = (-x, y) \]

**Rotation**  
\[ (x', y') = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) \]

**Enlargement**  
\[ (x', y') = (kx, ky) \]

\[ f(x) = x^2 \]

\[ f(x) = x^2 + c \]
Transformations: $f(x) \rightarrow f(x) = af(n(x - b)) + c$ \text{ or } (x, y) \rightarrow \left(\frac{x}{n} + b, ay + c\right)$

- Transformations of a function is one of the following:
  - Dilation (STRETCH) (from the $x$-axis or $y$-axis);
  - Reflection (FLIP) (in $x$-axis or $y$-axis);
  - Translation (SLIDE) (vertically and/or horizontally);
  - Rotation (we don’t study these).

- The order to deal with the transformations is DRT (alphabetical)
- The Cartesian Plane is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers.

Dilations
- This is a stretch or contraction of the graph from the $x$-axis or the $y$-axis
- $a$ causes a dilation of factor $a$ from the $x$-axis $(x, y) \rightarrow (x, ay)$
- $n$ causes a dilation of factor $\frac{1}{n}$ from the $y$-axis $(x, y) \rightarrow \left(\frac{x}{n}, y\right)$

- We describe the dilations like:
  - The graph is dilated by a factor of $a$ from the $x$-axis, or
  - The graph is dilated by a factor of $a$ parallel to the $y$-axis
  - The graph is dilated by a factor of $\frac{1}{n}$ from the $y$-axis

Example: Sketch the graph of $f(x) = 3x^2$ by comparing it to $f(x) = x^2$

<table>
<thead>
<tr>
<th>Here $a = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>First sketch $f(x) = x^2$</td>
</tr>
<tr>
<td>Then multiply each $y$ value by 3. ($= 3f(x)$)</td>
</tr>
<tr>
<td>The graph is dilated by a factor of 3 from the $x$-axis.</td>
</tr>
</tbody>
</table>

![3a Quadratic Function Transformation.ggb](https://example.com)

Example: Sketch $f(x) = (2x)^2$

<table>
<thead>
<tr>
<th>Here $n = _2_ $</th>
</tr>
</thead>
<tbody>
<tr>
<td>First sketch $f(x) = x^2$</td>
</tr>
<tr>
<td>Then multiply each $x$ value by $_\frac{1}{2}_$.</td>
</tr>
<tr>
<td>The graph is dilated by a factor of $\frac{1}{2}$ from the $y$-axis.</td>
</tr>
</tbody>
</table>

Could also be a dilation of factor 4 from the $x$-axis. Why?

![3a Quadratic Function Transformation.ggb](https://example.com)
Reflections

- There are three types of reflections:
  - In the $x$-axis, $y = -f(x)$, $(x, y) \rightarrow (x, -y)$
  - In the $y$-axis, $y = f(-x)$, $(x, y) \rightarrow (-x, y)$
  - In the line $y = x$, which we dealt with in Inverse functions.

Reflections in the $x$-axis, $y = -f(x)$ or when $a < 0$

Example: Sketch $f(x) = \frac{-x^2}{2}$

Here $a = -\frac{1}{2}$

The graph is reflected in the $x$-axis and dilated by a factor of $\frac{1}{2}$ from $x$-axis.

Reflections in the $y$-axis, $y = f(-x)$

Example: Sketch $f(x) = x^3 + 3$, $f(-x)$ and $-f(-x)$.

$$f(-x) = (-x)^3 + 3$$
$$= -x^3 + 3$$

$$-f(-x) = -(x^3 + 3)$$
$$= x^3 - 3$$

- $f(-x)$ is a reflection in both $x$ & $y$ axes.
For each of the following graphs of $y = f(x)$, sketch:

(i) $y = -f(x)$
(ii) $y = f(-x)$
(iii) $y = -f(-x)$
For each of the following graphs of \( y = f(x) \), sketch:

(i) \( y = -f(x) \)
(ii) \( y = f(-x) \)
(iii) \( y = -f(-x) \)
Translations
- There are two types of translations:
  - Along the direction of the $x$-axis: $f(x) = f(x-b) \ ; \ (x, y) \rightarrow (x + b, y)$
  - Along the direction of the $y$-axis: $f(x) = f(x) + c \ ; \ (x, y) \rightarrow (x, y + c)$

1. **Along the direction of the $x$-axis**: $f(x) = f(x-b)$
   - Sketch the graph of $f(x) = (x+4)^3$.
   - Translation of 4 units in the negative direction of the $x$-axis.

   ![GeoGebra](image1)

   - Sketch the graph of $f(x) = (x-2)^2$
   - _translation_ of _2_ units in the _positive_ direction of the X-axis.

   ![GeoGebra](image2)

2. **Along the direction of the $y$-axis**: $f(x) = f(x) + c$
   - Sketch the graph of $f(x) = x^2 + 4$
   - _translation_ of _4_ units in the _positive_ direction of the Y-axis.

   ![GeoGebra](image3)

“Repeated Factor Squared”
- Consider the function $f(x) = (x+3)(x-2)^2$
- The $X$– intercepts are -3 and 2
- (2, 0) is also a Turning Point
- “A repeated factor squared is both an $X$–Intercept and a Turning Point”
Repeated Factor Cubed

- Consider the function \( f(x) = (x + 1)^3(x - 4) \)
- The \( X \)-intercepts are -1 and 4
- (-1, 0) is also a Point of inflexion
- “A repeated factor cubed is both an \( X \)-Intercept and a Point of Inflection”

- **Ex4A Q 1, 3; Ex 3A Q 7 ab; Ex 3B Q 11a;**
- **Ex 3C Q 1; Ex 3D Q 1a; Ex 3E Q 2 abc**

Transformations Summary \( f(x) \to af(n(x - b)) + c \) or \( (x, y) \to \left(\frac{x}{n} + b, ay + c\right) \)

**Example 1:** State the transformations from \( f(x) \) to \( y = -2f(3(x + 4)) - 1 \).

A dilation of factor of 2 from the \( x \)-axis and a factor of \( \frac{1}{3} \) from the \( y \)-axis, followed by a reflection in the \( x \)-axis, then a translation of 4 units in the negative direction of the \( x \)-axis and a translation of 1 unit in the negative direction of the \( y \)-axis.

**Example 2:** Describe the transformations undergone by \( y = \log_e x \) to \( y = 1 - 3\log_e(2x - 8) \).

\[
y = 1 - 3\log_e(2x - 8) = -3\log_e(2(x - 4)) + 1
\]

A dilation of factor of 3 from the \( x \)-axis and a factor of \( \frac{1}{2} \) from the \( y \)-axis, followed by a reflection in the \( x \)-axis, then a translation of 4 units in the positive direction of the \( x \)-axis and a translation of 1 unit in the positive direction of the \( y \)-axis.

**Example 3:** Write the equation of the rule when \( y = x^2 \) is transformed by:
- a translation of 1 unit in the positive direction of the \( x \)-axis and 2 units in the positive direction of the \( y \)-axis, followed by,
- a dilation of factor of 2 from the \( y \)-axis, followed by,
- a reflection in the \( x \)-axis.

\[
\Rightarrow (x - 1)^2 + 2 \Rightarrow \left(\frac{x}{2} - 1\right)^2 + 2 \Rightarrow -\left(\frac{x}{2} - 1\right)^2 - 2 \quad \text{or} \quad y = -\left(\frac{1}{2}(x - 2)\right)^2 - 2
\]

\[
y = -\frac{1}{4}(x - 2)^2 - 2 \quad \text{or} \quad f(x) \to f(x - 1) + 2 \to f\left(\frac{x}{2} - 1\right) + 2 \to -f\left(\frac{x}{2} - 1\right) + 2 \to -f\left(\frac{x}{2} - 1\right) - 2
\]

Exercise on Sequence of Transformations

1. State the sequence of transformations that each of the following functions have undergone from \( y = f(x) \).

   (a) \( y = 3f(-2(x + 3)) + 4 \)
   (b) \( y = 0.5f(3(x - 2)) + 1 \)
   (c) \( y = 2f(-0.4(x + 3)) - 0.2 \)
   (d) \( y = 2 - 3f(2x + 1) \)

2. Describe the transformations undergone by each of the following functions to produce the second function.

   (a) \( y = \log_e x \) to \( y = 4\log_e(2x + 3) - 5 \)
   (b) \( y = \sqrt{x} \) to \( y = 2\sqrt{3x + 4} + 5 \)
   (c) \( y = \cos x \) to \( y = -3\cos\left(2x + \frac{\pi}{4}\right) + 1 \)
   (d) \( y = x^6 \) to \( y = 3(2x + 5)^6 - 2 \)
   (e) \( y = \sin x \) to \( y = 2\sin\pi(3x - 4) \)
Ex4E Q 1, 2, 3, 4; Ex 3A Q 7 d; Ex 3B Q 4; Ex 3C Q 2b, 4a; Ex 3D Q 4d; Ex 3E Q 1a; Ex4F Q 1, 2, 3, 4, 5, 6
Determining a Rule for a Function from a Graph

- **Worksheet** – Matching Graphs to their rules
  Match the following graphs with the correct equation:

  (a)         (b)  
  (c)         (d)  
  (e)        (f)  

A: )1(3
xxy
−
B :  )1( 2
xxy
−
C :  )1( 22
xxy
−
D: )1(2
xxy
−
E :  )1( 2
xxy
−
F :  )1( 22
xxy
−

\[
(a)=B, \ (b) = D, \ (c) = F, \ (d)= E , \ (e) = A, \ (f) = C
\]

- **Example:** Find the rule for:

\[
\begin{align*}
y &= y' \left(1-x\right) \\
y &= x(1-x^2) \\
y &= x^2 \left(1-x\right) \\
y &= x^2(1-x^2) \\
y &= x^2(1-x) \\
y &= x^2(1-x^2)
\end{align*}
\]

\[
\begin{align*}
(a)=B, \ (b) = D, \ (c) = F, \ (d)= E , \ (e) = A, \ (f) = C
\end{align*}
\]

- **Example:** Find the rule for:

\[
\begin{align*}
\text{We know the turning point: } (2,3) \therefore y &= a(x-b)^2 + c \\
y &= a(x-2)^2 + 3 \\
y - \text{int: } (0,11) \Rightarrow 11 &= a(0-2)^2 + 3 \\
11 &= 4a + 3 \\
8 &= 4a \\
2 &= a \\
\therefore y &= 2(x-2)^2 + 3 \\
or \ y &= 2x^2 - 8x + 11
\end{align*}
\]
• Example: Find the rule for:

We know the x-ints: \((-1,0),(4,0)\) \(\therefore y = a(x + 1)(x - 4)\)

Use \((0,-6) \Rightarrow -6 = a(0 + 1)(0 - 4)\)

\(-6 = a \cdot 1 \cdot -4\)

\(-6 = -4a\)

\(\frac{3}{2} = a\)

\(\therefore y = \frac{3}{2} (x + 1)(x - 4)\)

• Graphical Calculator can be used for this example.
  - Insert Lists & Spreadsheet
  - X-values – List1
  - Y-values – List2
  - Regression – Menu – Statistics - Calculations

• Example: Find the rule for: \((1, -1)\) on curve.

- This is a quartic.
- Point of inflection at the origin, therefore a repeated factor cubed, \(x^3\)
- Also an x-intercept at \((2,0)\)

\[y = ax^3 (x - 2)\]

\((1,-1)\)

\(-1 = a(1)^3 (1 - 2)\)

\(-1 = -a\)

\(1 = a\)

\(\therefore y = x^3 (x - 2)\)

• Example: Find the rule for:

\[y = a(x - 1)(x + 3)^2\]

\((0,-9) \Rightarrow -9 = a(0 - 1)(0 + 3)^2\)

\(-1 = a \Rightarrow y = -(x - 1)(x + 3)^2\)

• Ex4A Q 8, 9; Ex4B Q 1, 2, 3, 4, 7, 8, 9; Ex4G Q 1, 3, 4, 5, 6, 7, 8; Ex3G 3, 4, 5, 6, 7ab, 8, 9
Transformations of $f(x) = x^p; p = -1, -3, \ldots$

$\frac{1}{x^p} \rightarrow \frac{a}{(n(x - b))^p} + c$ or $a f(n(x - b)) + c$

Examples: Sketch the graph of:

(i) $f(x) = \frac{2}{x}$

(ii) $f(x) = \frac{1}{2x^3}$

(iii) $f(x) = \frac{4}{x - 2}$

(iv) $f(x) = \frac{4}{2 - x}$

(v) First show that $f(x) = \frac{-3x - 3}{x + 2}$ is equal to $f(x) = \frac{3}{x + 2} - 3$

(i) Dilation of factor of 2 from the $x$-axis:

$a = 2 \rightarrow 2 f(x)$

(ii) Dilation of factor $\frac{1}{2}$ from the $x$-axis:

$a = \frac{1}{2} \rightarrow \frac{1}{2} f(x)$

(iii) A dilation of factor 4 from the $x$-axis and a translation of 2 units to the right (positive direction of the $x$-axis):

Note: the vertical asymptote also moves 2 units to the right.

$Y$-intercept:

$x = 0$

$y = \frac{4}{0 - 2} = -2$

$a = 4 \& b = 2 \rightarrow 4 f(x - 2)$
(iv) Re-write the function: \( f(x) = \frac{4}{-x + 2} = \frac{4}{-(x-2)} \)

dilation of factor 4, a reflection in the y-axis and a translation of 2 units to the right (positive direction of the x-axis):

Note: the vertical asymptote also moves 2 units to the right. Y-Intercept: \( y = 2 \)

\[ a = 4, b = 2 \text{ & } n = -1 \rightarrow 4f(-(x-2)) \]
\[ f(x) = \frac{-3x - 3}{x + 2} = \frac{-3(x + 2) + 3}{x + 2} = \frac{-3(x + 2) + 3}{x + 2} = \frac{3}{x + 2} - 3 \]

A dilation of factor 3 from the \(x\)-axis, a translation of three units in the negative direction of the \(y\)-axis and a translation of 2 units in the negative direction of the \(x\)-axis:

\[
\begin{align*}
  y &= 0 \\
  \frac{3}{x + 2} - 3 &= 0 \\
  \frac{3}{x + 2} &= 3 \\
  1 &= x + 2 \\
  -1 &= x
\end{align*}
\]

Intercepts:
\[
\begin{align*}
  x &= 0 \\
  y &= \frac{3}{0 + 2} - 3 \\
  y &= \frac{3}{2} - 3 \\
  y &= \frac{-3}{2} \\
  y &= \frac{-3}{2}
\end{align*}
\]

\[ a = 3, b = \frac{-2}{2} \text{ and } c = \frac{-3}{2} \rightarrow 3f(x + 2) - 3 \]

- **Ex3A** Q 2, 3 aefghk, 4, 5 be, 8b; **Ex3B** Q 1, 5 ab, 7 **Ex3D** Q 4 c; **Ex3E** Q 4af; **Ex3F** 1 abef, 2 dgij, 3, 4, 5 ab

- Hint Ex 3F Q4
  \[
  y = \frac{4x + 5}{2x + 3} = \frac{2(2x + 3) - 1}{2x + 3} = 2 - \frac{1}{2x + 3}
  \]
- Or synthetic division
  \[
  y = \frac{4x + 5}{2x + 3} = \frac{2x + \frac{5}{2}}{x + \frac{3}{2}} \text{ first (divide all terms by 2)}
  \]
Transformations of \[ f(x) = x^p; \ p = -2, -4, \ldots \]

\[ \frac{1}{x^p} \rightarrow \frac{a}{(n(x-b))^p} + c \]

**Examples:** Sketch the graph of:

(i) \[ f(x) = \frac{2}{x^2} \]
(ii) \[ f(x) = \frac{1}{2x^4} \]
(iii) \[ f(x) = \frac{4}{(x-2)^3} \]
(iv) \[ f(x) = \frac{-3}{(x+2)^3} \]
(v) \[ f(x) = \frac{3}{(x+2)^3} - 3 \]

**Note:**

if \[ g(x) = \frac{1}{x^2} \rightarrow f(x) = 2g(x) \]

if \[ h(x) = \frac{1}{x^4} \rightarrow f(x) = \frac{1}{2}h(x) \]

(iii) dilation of factor 4 and a translation of 2 units to the right (positive direction of the x-axis): Note: the vertical asymptote also moves 2 units to the right.

\[ x = 0 \]

\[ Y\text{-intercept: } y = \frac{4}{(0-2)^2} = \frac{4}{(-2)^2} = \frac{4}{4} = 1 \]
if \( g(x) = \frac{1}{x^2} \rightarrow f(x) = 4g(x - 2) \)

(iv) dilation of factor 3, a reflection in the \( x \)-axis and a translation of 2 units to the left (negative direction of the \( x \)-axis):

Note: the vertical asymptote also moves 2 units to the left.

\[ y = \frac{-3}{(0+2)^2} = \frac{-3}{4} \]

if \( g(x) = \frac{1}{x^2} \rightarrow f(x) = -3g(x + 2) \)

(v) dilation of factor 3, a translation of three units down (negative direction of the \( y \)-axis) and a translation of 2 units to the left (negative direction of the \( x \)-axis):

\[ y = 0 \]
\[ \frac{3}{(x+2)^2} = 3 \]

Intercepts:
\[ 3 = 3(x+2)^2 \]
\[ 1 = (x+2)^2 \]
\[ \pm 1 = x + 2 \]
\[ x = -3, -1 \]

if \( g(x) = \frac{1}{x^2} \rightarrow f(x) = 3g(x + 2) - 3 \)

- **Ex3A** Q 3 bci, 5cd, 7c, 8a; **Ex3B** Q 2, 5d, 6, 10 ae, 11 bde; **Ex3C** Q 4 cd; **Ex3D** Q 1c, 4 efg, 6; **Ex3E** Q 1 b, 2 cd, 3a, 4bc; **Ex3F** Q cdg, 2h, 5c
Transformations of functions of the form \( f(x) = x^q \)

\[
x^q \rightarrow a(n(x - b))^{\frac{p}{q}} + c \quad \text{OR} \quad x^q \rightarrow a\sqrt[q]{[n(x - b)]^p} + c
\]

- **Ex3A** Q 6, 7c; **Ex3B** Q 3, 5c, 8, 9, 10 bcd, 11 cfg; **Ex 3C** Q 2a, 3, 4 befg; **Ex3D** Q 1b, 4b, 5, 7; **Ex3E** Q 1, 3 de, 4de; **Ex3F** Q 2 abcef, 5 def
Determining rules for \( f(x) = x^n \)

Example: It is known that the points (1, 5) and (4, 2) lie on a curve with the equation \( y = \frac{a}{x} + b \).
Find the values of \( a \) and \( b \).

Solution:
\[
\begin{align*}
\text{When } x &= 1, y = 5 \implies 5 = a + b \quad (1) \\
\text{When } x &= 4, y = 2 \implies 2 = \frac{a}{4} + b \quad (2)
\end{align*}
\]

Subtract (2) from (1): \( 3 = \frac{3a}{4} \)

\( \therefore a = 4 \)

Substitute in (1): \( 5 = 4 + b \)

\( b = 1 \)

\( y = \frac{a}{x} + b \)

Example 2: It is known that the points (2, 1) and (10, 6) lie on a curve with equation \( y = a\sqrt{x-1} + b \). Find the equation.

Solution:
\[
\begin{align*}
(2, 1): & \quad 1 = a\sqrt{2-1} + b \\
& \quad 1 = a + b \quad (1) \\
(10, 6): & \quad 6 = a\sqrt{10-1} + b \\
& \quad 6 = 3a + b \quad (2)
\end{align*}
\]

Subtract (1) from (2): \( 5 = 2a \)

\( \therefore a = \frac{5}{2} \)

Substitute in (1): \( 1 = \frac{5}{2} + b \)

\( b = -\frac{3}{2} \)

\( \therefore y = \frac{5}{2}\sqrt{x-1} - \frac{3}{2} \) or \( y = \frac{5\sqrt{x-1} - 3}{2} \)

- Ex3H Q 1, 2, 3, 4, 5, 6, 7, 8
Transformations using Matrices:
- \((x', y')\) is called the image of \((x, y)\).
- the transformations are written as follows:

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \begin{bmatrix}
  x \\
  y
\end{bmatrix} + \begin{bmatrix}
  e \\
  f
\end{bmatrix}
\]

- You can have more than one dilation/reflection matrix.
- Remember: multiply rows by columns, add/subtract elements in the same position.
- The transformation matrices are:

<table>
<thead>
<tr>
<th>Transformation Type</th>
<th>Matrix</th>
<th>Effect</th>
</tr>
</thead>
</table>
| Reflection in the \(x\)-axis               | \[
\begin{bmatrix}
  1 & 0 \\
  0 & -1
\end{bmatrix}
\] | \((x, y) \rightarrow (x, -y)\) \(f(x) \rightarrow -f(x)\) |
| Reflection in the \(y\)-axis               | \[
\begin{bmatrix}
  -1 & 0 \\
  0 & 1
\end{bmatrix}
\] | \((x, y) \rightarrow (-x, y)\) \(f(x) \rightarrow f(-x)\) |
| Reflection in the line \(y=x\)             | \[
\begin{bmatrix}
  0 & 1 \\
  1 & 0
\end{bmatrix}
\] | \((x, y) \rightarrow (y, x)\) \(f(x) \rightarrow f(y)\) |
| Dilation of factor \(a\) from the \(x\)-axis| \[
\begin{bmatrix}
  1 & 0 \\
  0 & a
\end{bmatrix}
\] | \((x, y) \rightarrow (x, ay)\) \(f(x) \rightarrow af(x)\) |
| Dilation of factor \(k\) from the \(y\)-axis (note \(n = 1/k\)) | \[
\begin{bmatrix}
  k & 0 \\
  0 & 1
\end{bmatrix}
\] | \((x, y) \rightarrow (kx, y)\) \(f(x) \rightarrow f\left(\frac{x}{k}\right)\) |
| Translation Matrix (add)                   | \[
\begin{bmatrix}
  b \\
  c
\end{bmatrix}
\] | \((x, y) \rightarrow (x + b, y + c)\) \(f(x) \rightarrow f(x - b) + c\) |

**Example 1:** find the image of the point \((2, 3)\) under:
- a reflection in the \(x\)-axis
- a dilation of factor 4 from the \(y\)-axis

\[
\begin{bmatrix}
  1 & 0 \\
  0 & -1
\end{bmatrix} \begin{bmatrix}
  2 \\
  3
\end{bmatrix} = \begin{bmatrix}
  2 \\
  -3
\end{bmatrix} \Rightarrow (2, 3) \rightarrow (2, -3)
\]

\[
\begin{bmatrix}
  4 & 0 \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  2 \\
  3
\end{bmatrix} = \begin{bmatrix}
  8 \\
  3
\end{bmatrix} \Rightarrow (2, 3) \rightarrow (8, 3)
\]

**Example 2:** Consider a linear transformation such that \((1, 0) \rightarrow (3, -1)\) and \((0, 1) \rightarrow (-2, 4)\). Find the image of \((-3, 5)\)

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \begin{bmatrix}
  1 \\
  0
\end{bmatrix} = \begin{bmatrix}
  3 \\
  -1
\end{bmatrix} \text{ and } \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \begin{bmatrix}
  0 \\
  1
\end{bmatrix} = \begin{bmatrix}
  -2 \\
  4
\end{bmatrix}
\]

\[
\Rightarrow a = 3, c = -1 \quad b = -2, d = 4
\]

\[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix} = \begin{bmatrix}
  3 & -2 \\
  -1 & 4
\end{bmatrix} \begin{bmatrix}
  -3 \\
  5
\end{bmatrix} = \begin{bmatrix}
  -19 \\
  23
\end{bmatrix} \Rightarrow (-3, 5) \rightarrow (19, -23)
\]
Example 3: A transformation is defined by the matrix \[
\begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix}
\]. Find the equation of the graph of \( y = \sin(x) + x \), under this transformation.

Solution:

1. Write the dilations in terms of matrices

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
0 & 3
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix}
\]

2. Multiply matrices

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
2 \times x + 0 \times y \\
0 \times x + 3 \times y
\end{bmatrix} = \begin{bmatrix}
2x \\
3y
\end{bmatrix}
\]

3. Determine the result in terms of \( x' \) and \( y' \) & rearrange to make \( x \) and \( y \) the subject of each equation.

\[
x' = 2x \quad \Rightarrow \frac{x'}{2} = x \\
y' = 3y \quad \Rightarrow \frac{y'}{3} = y
\]

4. Sub each into the original equation.

\[
y' = 3 \sin \left( \frac{x'}{2} \right) + \frac{x'}{2}
\]

5. Rearrange to make \( y' \) the subject

\[
y' = 3 \sin \left( \frac{x'}{2} \right) + \frac{3x'}{2}
\]

6. Then drop the ‘

\[
y = 3 \sin \left( \frac{x}{2} \right) + \frac{3x}{2}
\]

Example 4: A transformation is described by the matrix equation \( A(X + B) = X' \), where

\[
A = \begin{bmatrix}
0 & -3 \\
2 & 0
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
1 \\
2
\end{bmatrix}
\]

Find the image of the straight line with equation \( y = 2x + 5 \) under this transformation.

\[
A(X + B) = X'
\]

\[
\begin{bmatrix}
0 & -3 \\
2 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -3 \\
2 & 0
\end{bmatrix} \begin{bmatrix}
x + 1 \\
y + 2
\end{bmatrix} = \begin{bmatrix}
x' \\
y'
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3(y+2) \\
2(x+1)
\end{bmatrix} = \begin{bmatrix}
x' \\
y'
\end{bmatrix} \quad \Rightarrow x' = -3(y+2) \quad \text{&} \quad y' = 2(x+1)
\]

\[
y = \frac{-x'}{3} - 2 \quad \text{&} \quad x = \frac{y'}{2} - 1
\]

Sub. into \( y = 2x + 5 \)...

\[
\frac{-x'}{3} - 2 = 2 \left( \frac{y'}{2} - 1 \right) + 5
\]

\[
y' = \frac{-x'}{3} - 5
\]

Ex3I Q 1, 3, 4, 6, 7, 8, 9, 10, 11, 13, 15, 16
### Transformations of standard graphs

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>Parent Graph</th>
<th>$af(n(x-b)) + c$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td><img src="image1" alt="Graph" /></td>
<td>$ax + c$</td>
<td><img src="image2" alt="Example Graph" /></td>
</tr>
<tr>
<td>$x^2$</td>
<td><img src="image3" alt="Graph" /></td>
<td>$a(n(x-b))^2 + c$</td>
<td><img src="image4" alt="Example Graph" /></td>
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<tr>
<td>$a(x-m)(x-n)$</td>
<td><img src="image5" alt="Graph" /></td>
<td>$ax^2 + bx + c$</td>
<td><img src="image6" alt="Example Graph" /></td>
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<tr>
<td>$x^3$</td>
<td><img src="image7" alt="Graph" /></td>
<td>$a(n(x-b))^3 + c$</td>
<td><img src="image8" alt="Example Graph" /></td>
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<tr>
<td>$x^4$</td>
<td><img src="image9" alt="Graph" /></td>
<td>$a(n(x-b))^4 + c$</td>
<td><img src="image10" alt="Example Graph" /></td>
</tr>
<tr>
<td>$\frac{1}{x}$ or $x^{-1}$</td>
<td><img src="image11" alt="Graph" /></td>
<td>$\frac{a}{n(x-b)} + c$</td>
<td><img src="image12" alt="Example Graph" /></td>
</tr>
<tr>
<td>Function</td>
<td>Graph</td>
<td>Function</td>
<td>Graph</td>
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<tr>
<td>$\frac{1}{x^2}$ or $x^{-2}$</td>
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<td>$\frac{a}{(n(x-b))^2} + c$</td>
<td><img src="image2.png" alt="Image" /></td>
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<tr>
<td>$\frac{1}{x}$ or $\sqrt{x}$</td>
<td><img src="image3.png" alt="Image" /></td>
<td>$a\sqrt{n(x-b)} + c$</td>
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<tr>
<td>$\frac{p}{x^2}$ or $\sqrt[n]{x^p}$</td>
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<td>$a^{\sqrt[n]{n(x-b)^p}} + c$</td>
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<td>$ae^{n(x-b)} + c$</td>
<td><img src="image8.png" alt="Image" /></td>
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<tr>
<td>$m^x$</td>
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<td>$am^{n(x-b)} + c$</td>
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<td>$\log_e x$</td>
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<td>$a\log_e (n(x-b)) + c$</td>
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<tr>
<td>Function</td>
<td>Graph 1</td>
<td>Graph 2</td>
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<tr>
<td>( \log_{10} x )</td>
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<tr>
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</tr>
<tr>
<td>( \tan x )</td>
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<td>( a \tan(n(x - b)) + c )</td>
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\[ y = x^q \]

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<tr>
<th>$p$</th>
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<th>Example graph</th>
<th>Equations</th>
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<td>$p &gt; q$</td>
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<td>$q$ odd</td>
<td>$R$</td>
<td>$y = x^3$</td>
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<tr>
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<td>$x \geq 0$</td>
<td>$R$</td>
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<tr>
<td>$p$ even</td>
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<td>$y = x^{3/2}$</td>
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<tr>
<td>&lt; $q$</td>
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<td>$q$ odd</td>
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<td>$R$</td>
<td>$y = x^3$</td>
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</table>
VCAA EXAM QUESTIONS for TRANSFORMATIONS

2008

Question 8
The graph of the function \( f: D \to \mathbb{R}, \ f(x) = \frac{x - 3}{2 - x} \) where \( D \) is the maximal domain has asymptotes
A. \( x = 3, \ y = 2 \)
B. \( x = -2, \ y = 1 \)
C. \( x = 1, \ y = -1 \)
D. \( x = 2, \ y = -1 \)
E. \( x = 2, \ y = 1 \)

\[
Q8 \quad f(x) = \frac{x - 3}{2 - x} = \frac{(x - 3)}{(x - 2)} = \frac{(x - 2 - 1)}{(x - 2)} = \left(1 - \frac{1}{x-2}\right)
\]
\[
= \frac{1}{x-2} - 1. \text{ Asymptotes: } x = 2, \ y = -1. \quad D
\]

Question 2
On the axes below, sketch the graph of \( f: \mathbb{R} \to \mathbb{R}, \ f(x) = 2 - \frac{4}{x+1} \)
Label all axis intercepts. Label each asymptote with its equation.

Question 2
Marks

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</table>

This question was generally well done with most students recognising a hyperbola. Students should label asymptotes and/or axial intercepts. In this question students sometimes gave the wrong intercept (\( x = 3 \) was common), extra intercepts or no horizontal asymptote.

2009

Question 2
At the point \( (1, 1) \) on the graph of the function with rule \( y = (x - 1)^3 + 1 \)
A. there is a local maximum.
B. there is a local minimum.
C. there is a stationary point of inflection.
D. the gradient is not defined.
E. there is a point of discontinuity.

Question 21
A cubic function has the rule \( y = f(x) \). The graph of the derivative function \( f' \) crosses the \( x \)-axis at \( (2, 0) \) and \( (-3, 0) \). The maximum value of the derivative function is 10.
The value of \( x \) for which the graph of \( y = f(x) \) has a local maximum is
A. \(-2\)
B. \(2\)
C. \(-3\)
D. \(3\)
E. \(\frac{1}{2}\)

Q21 \quad f'(x) = a(x-2)(x+3) \) is a quadratic function. For it to have a maximum value, \( a < 0 \).
2011

Question 3

a. Consider the function \( f: \mathbb{R} \to \mathbb{R}, f(x) = 4x^3 + 5x - 9 \).
   i. Find \( f'(x) \)

   \[ f'(x) = 12x^2 + 5 \]

   1 + 1 = 2 marks

   ii. Explain why \( f'(x) \geq 5 \) for all \( x \).

   1 + 1 = 2 marks

b. The cubic function \( p \) is defined by \( p: \mathbb{R} \to \mathbb{R}, p(x) = ax^3 + bx^2 + cx + k \), where \( a, b, c \) and \( k \) are real numbers.
   i. If \( p \) has \( m \) stationary points, what possible values can \( m \) have?

   1 + 1 = 2 marks

   ii. If \( p \) has an inverse function, what possible values can \( m \) have?

   1 + 1 = 2 marks

c. The cubic function \( q \) is defined by \( q: \mathbb{R} \to \mathbb{R}, q(x) = 3 - 2x^3 \).
   i. Write down an expression for \( q^{-1}(x) \).

   2 + 2 = 4 marks
d. The cubic function $g$ is defined by $g: R \rightarrow R, g(x) = x^3 + 2x^2 + cx + k$, where $c$ and $k$ are real numbers.

i. If $g$ has exactly one stationary point, find the value of $c$.

\[ g(x) = x^3 + 2x^2 + cx + k \]
\[ g'(x) = 3x^2 + 4x + c \]

For $g'(x) = 0$ at exactly one point, $\Delta = 4^2 - 4(3)c = 0$, so $c = \frac{4}{3}$.

Q3dii $g(x) = x^3 + 2x^2 + \frac{4}{3}x + k$. Let $g'(x) = 3x^2 + 4x + \frac{4}{3} = 0$.

The stationary point is at $x = -\frac{2}{3}$.

At the intersection of $y = g(x)$ and $y = g^{-1}(x)$, $y = x$.

\[ x^3 + 2x^2 + \frac{4}{3}x + k = x \]
\[ \left(-\frac{2}{3}\right)^3 + 2\left(-\frac{2}{3}\right)^2 + \frac{4}{3}\left(-\frac{2}{3}\right) + k = -\frac{2}{3} \]

so $k = \frac{-10}{27}$.

---

2012

**Question 8**

The function $f: R \rightarrow R, f(x) = ax^3 + bx^2 + cx$, where $a$ is a negative real number and $b$ and $c$ are real numbers.

For the real numbers $p < m < 0 < n < q$, we have $f(p) = f(q) = 0$ and $f(m) = f(n) = 0$.

The gradient of the graph of $y = f(x)$ is negative for

- A. $(\infty, m) \cup (n, \infty)$
- B. $(m, n)$
- C. $(p, 0) \cup (q, \infty)$
- D. $(p, m) \cup (0, q)$
- E. $(p, q)$

<table>
<thead>
<tr>
<th>Question</th>
<th>% A</th>
<th>% B</th>
<th>% C</th>
<th>% D</th>
<th>% E</th>
<th>% No answer</th>
</tr>
</thead>
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<td>49</td>
<td>14</td>
<td>18</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

The gradient of the graph of $y = f(x)$ is negative for $(\infty, m) \cup (n, \infty)$. There is a local minimum at $x = m$ and a local maximum at $x = n$.

**Question 16**

The graph of a cubic function $f$ has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$.

The values of $c$, such that the equation $f(x) + c = 0$ has exactly one solution, are

- A. $3 < c < 8$
- B. $c > -3$ or $c < -8$
- C. $-8 < c < -3$
- D. $c < 3$ or $c > 8$
- E. $c < -8$

<table>
<thead>
<tr>
<th>Question</th>
<th>16</th>
<th>13</th>
<th>17</th>
<th>27</th>
<th>84</th>
<th>8</th>
<th>1</th>
</tr>
</thead>
</table>

The local maximum will be touching the $x$-axis when $c = 3$, giving two distinct solutions. So if $c < 3$ there will be one solution. The local minimum will be above the $x$-axis when $c > 8$. Hence $c < 3$ or $c > 8$.

2014

**Question 1**

The point $P(4, -3)$ lies on the graph of a function $f$. The graph of $f$ is translated four units vertically up and then reflected in the $y$-axis.

The coordinates of the final image of $P$ are

- A. $(4, -3)$
- B. $(-4, 3)$
- C. $(0, -3)$
- D. $(4, -6)$
- E. $(-4, -1)$

Q1 $(4, -3) \rightarrow (4, 1) \rightarrow (-4, 1)$
Question 12
The transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) with rule
\[
T \left( \begin{array}{c} x \\ y \end{array} \right) = \left[ \begin{array}{cc} -1 & 0 \\ 0 & 2 \end{array} \right] \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} 1 \\ 2 \end{array} \right)
\]
maps the line with equation \( x - 2y = 3 \) onto the line with equation
A. \( x + y = 0 \)
B. \( x + 4y = 0 \)
C. \( -x - y = 4 \)
D. \( x + 4y = -6 \)
E. \( x - 2y = 1 \)

\[ Q12 \quad T \left( \begin{array}{c} x \\ y \end{array} \right) = \left[ \begin{array}{c} -x + 1 \\ 2y - 2 \end{array} \right] = \left[ \begin{array}{c} x' \\ y' \end{array} \right] \Rightarrow x = 1 - x' \quad \text{and} \quad y = \frac{y' + 2}{2}
\]
\[ \therefore x - 2y = 3 \rightarrow -x' - y' = 4 \]

Question 5 (13 marks)
Let \( f : \mathbb{R} \to \mathbb{R}, f(x) = (x - 3)(x - 1)(x^2 + 3) \) and \( g : \mathbb{R} \to \mathbb{R}, g(x) = x^3 - 8x \).

a. Express \( x^3 - 8x \) in the form \( x(x - a)(x + b)^2 + c \).  

b. Describe the translation that maps the graph of \( y = f(x) \) onto the graph of \( y = g(x) \).

\[ \begin{align*}
Q5a \quad g(x) &= x^3 - 8x = x(x^3 - 2^3) = x(x - 2)(x^2 + 2x + 4) \\
&= x(x - 2)(x^2 + 2x + 1 + 3) = x(x - 2)((x + 1)^2 + 3)
\end{align*} \]

Q5b Translate the graph of \( y = f(x) \) to the left by 1. 
\[ g(x) = f(x + 1) = (x + 1 - 3)(x + 1 - 1)((x + 1)^2 + 3) = x(x - 2)((x + 1)^2 + 3) \]

Q5ci For one positive x-intercept, \( d - 1 \geq 0 \) and \( d - 3 < 0 \), i.e. \( 1 \leq d < 3 \)
\[ \therefore d < 3 \]

Q5cii For two positive x-intercepts, \( d - 1 < 0 \) and \( d - 3 < 0 \), \( \therefore d < 1 \) and \( d < 3 \), \( \therefore d < 1 \)

Q5d The graphs of \( y = g(x) = x^3 - 8x \) and \( y = n \) intersect at one point when \( y = n \) is a tangent to \( y = g(x) \) at its minimum turning point. 
\[ g'(x) = 3x^2 - 8 \]
Let \( g'(x) = 0 \), \( \therefore x = \pm \sqrt[3]{\frac{8}{3}} \) and \( n = g\left( \sqrt[3]{\frac{8}{3}} \right) = -\frac{6 \times \sqrt[3]{\frac{8}{3}}}{3} \)
e. At the point \((u, g(u))\), the gradient of \(y = g(x)\) is \(m\) and at the point \((v, g(v))\), the gradient is \(-m\), where \(m\) is a positive real number.

i. Find the value of \(u^3 + v^3\).

\[\begin{align*}
\text{At } (u, g(u)), \quad g'(u) &= m, \quad \therefore 4u^3 - 8 = m \\
\text{At } (v, g(v)), \quad g'(v) &= -m, \quad \therefore 4v^3 - 8 = -m \\
\therefore 4u^3 + 4v^3 - 16 &= 0, \quad \therefore u^3 + v^3 = 4
\end{align*}\]

ii. Solve \(u^3 + v^3 = 4\) and \(u + v = 1\) simultaneously by CAS.

\[
\begin{align*}
\text{Given } g'(u) &= m \text{ is positive, and } g'(v) &= -m \text{ is negative, } \therefore u > v \\
\text{Hence } u &= \frac{1 + \sqrt{5}}{2} \quad \text{and } v &= \frac{1 - \sqrt{5}}{2}.
\end{align*}
\]

f. i. Find the equation of the tangent to the graph of \(y = g(x)\) at the point \((p, g(p))\). 1 mark

\[
\begin{align*}
\text{Equation of the tangent: } y &= g'(p)(x - p) \\
&= (4p^3 - 8)(x - p)
\end{align*}
\]

ii. Find the equations of the tangents to the graph of \(y = g(x)\) that pass through the point with coordinates \(\left(\frac{3}{2}, -12\right)\). 3 marks

\[
\begin{align*}
\text{Equation of the tangents: } y &= (4p^3 - 8)x - 3p^4 \\
\therefore 3p^4(p - 2) &= 0, \quad \therefore p = 0 \text{ or } p = 2 \\
\text{The tangents are: } y &= -8x \text{ and } y = 24x - 48
\end{align*}
\]
2015

Question 3

The rule for a function with the graph above could be
A. \( y = -2(x + b)(x - c)^2(x - d) \)
B. \( y = 2(x + b)(x - c)^2(x - d) \)
C. \( y = -2(x - b)(x - c)^2(x - d) \)
D. \( y = 2(x - b)(x - c)(x - d) \)
E. \( y = -2(x - b)(x + c)^2(x + d) \)

The rule for the graph is in the form \( f(x) = a(x - b)(x - c)^2(x - d) \), where \( a \) is negative and could be \(-2\).

\( f(x) = -2(x - b)(x - c)^2(x - d) \)

\( a \) is negative; for example if \( b = -2 \), the factor is \((x - (-2)) = (x + 2)\).

Most students chose option A. \( y = -2(x + b)(x - c)^2(x - d) \), but the factor \((x + b)\) is incorrect.

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</table>

Question 11

The transformation that maps the graph of \( y = \sqrt{8x^2 + 1} \) onto the graph of \( y = \sqrt{x^2 + 1} \) is a
A. dilation by a factor of 2 from the y-axis.
B. dilation by a factor of 2 from the x-axis.
C. dilation by a factor of \( \frac{1}{2} \) from the x-axis.
D. dilation by a factor of 8 from the y-axis.
E. dilation by a factor of \( \frac{1}{2} \) from the y-axis.

\[ y_1 = \sqrt{8(x+1)^2 + 1} \]

The graph of \( y_1 \) has been dilated by a factor of 2 from the y-axis to get the graph of \( y_1 \).

This can be shown by sketching the graphs of both functions. For example, the point with coordinates \((1, 3)\) is transformed to \((2, 3)\).

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Question 17

A graph with rule \( f(x) = x^2 - 3x^2 + c \), where \( c \) is a real number, has three distinct x-intercepts.

The set of all possible values of \( c \) is
A. \( \mathbb{R} \)
B. \( \mathbb{R}^2 \)
C. \((0, 4)\)
D. \((0, 4)\)
E. \((0, 4)\)

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</table>

Question 20

If \( f(x - 1) = x^2 - 2x + 3 \), then \( f(x) \) is equal to
A. \( x^2 - 2 \)
B. \( x^2 + 2 \)
C. \( x^2 - 2x + 2 \)
D. \( x^2 - 2x + 4 \)
E. \( x^2 - 4x + 6 \)

\( B \)