Ejercicios - unidad 14.7
Integrales triples en Coordenadas cilíndricas y esféricas.

En los ejercicios del 1 a 6, evaluar la integral iterada.

1) $\int_{-1}^{5}\int_{0}^{\frac{\pi}{2}}\int_{0}^{3} r\cos\theta \, dr\,d\theta\,dz$

$I = \int_{-1}^{5} dz \cdot \int_{0}^{\frac{\pi}{2}} \cos\theta \, d\theta \cdot \int_{0}^{3} r\,dr$

$= z\Big|_{-1}^{5} \cdot \operatorname{sen}\theta\Big|_{0}^{\pi/2} \cdot \frac{r^2}{2}\Big|_{0}^{3}$

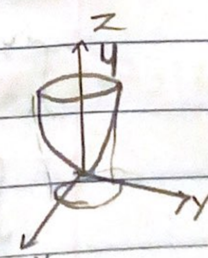
$= (5)+1 \cdot (\operatorname{sen}(\pi/2) - \operatorname{sen}(0)) \cdot \frac{(3)^2}{2} - \frac{(0)^2}{2}$

$= (6)(1-0)\cdot \frac{9}{2}$

$= 27$

En los ejercicios 13 a 16, Convertir la integral de Coordenadas rectangulares a Coordenadas cilíndricas y a Coordenadas esféricas, y evaluar la integral iterada más sencilla.

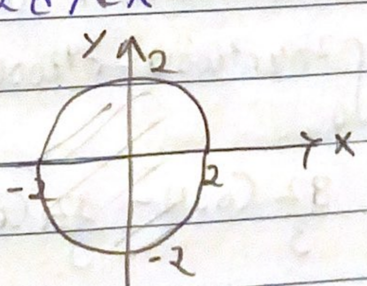
13) $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} x \, dz\,dy\,dx$



R.
$x^2+y^2 \le z \le 4$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$-2 \le x \le 2$



en R: (cilindrica)

$$r^2 \le z \le 4$$
$$0 \le r \le 2$$
$$0 \le \theta \le 2\pi$$

$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r\cos\theta \, r \, dz\,dr\,d\theta$$

R (esferica)

$$0 \le \rho \le 4\sec\phi \qquad 0 \le \rho \le \cot\phi\csc\phi$$
$$0 \le \phi \le \tan^{-1}(\tfrac{1}{2}) \quad + \quad \tan^{-1}(\tfrac{1}{2}) \le \phi \le \pi/2$$
$$0 \le \theta \le 2\pi \qquad 0 \le \theta \le 2\pi$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\sec\phi} \rho^2 \operatorname{sen}^2\phi \, \rho\cos\theta \, d\rho\,d\phi\,d\theta$$

$$+\int_0^{2\pi}\int_{\tan^{-1}(\frac{1}{2})}^{\pi/2}\int_0^{\cot\phi\csc\phi} \rho\cos\theta \, \rho^2 \operatorname{sen}^2\phi \, d\rho\,d\phi\,d\theta$$

Resolver:

$$\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r^2\cos\theta \, dz\,dr\,d\theta$$

$$\int_{r^2}^{4} r^2\cos\theta\,dz = r^2\cos\theta \left[z\right]_{r^2}^{4} = 4(r^2\cos\theta) - r^2(r^2\cos\theta)$$

$$= \int_0^{2} 4r^2\cos\theta - r^4\cos\theta \, dr = \frac{4}{3}\cos\theta\, r^3 - \frac{r^5}{5}\cos\theta \Big|_0^{2}$$

$$= \frac{32}{3}\cos\theta - \frac{32}{5}\cos\theta = \cos\theta\left(\frac{32}{3} - 32/5\right)$$
$$= \cos\theta \cdot \frac{64}{15}$$

$$\int_0^{2\pi} \frac{64}{15} \cos\theta \, d\theta$$

$$= \frac{64}{15} (\operatorname{sen}\theta) \Big|_0^{2\pi} = \frac{64}{15} (\operatorname{sen}(2\pi) - \operatorname{sen}(0))$$

$$= \frac{64}{15} (0) = \boxed{0}$$

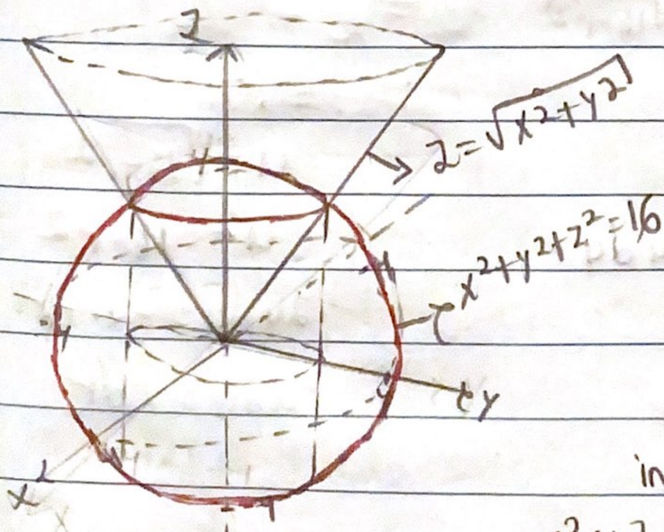
Volumen. En los ejercicios 17 a 32, utilizar coordenadas cilíndricas para hallar el volumen del sólido.

18) Sólido interior a $x^2+y^2+z^2=16$ y exterior a $z=\sqrt{x^2+y^2}$

$$V = \iiint_G dV$$

$$z^2 = x^2+y^2$$

$$0 = x^2+y^2-z^2$$



R: (cilíndricas) =

$$-\sqrt{16-r^2} \le z \le r$$

$$0 \le r \le 2\sqrt{2}$$

$$0 \le \theta \le 2\pi$$

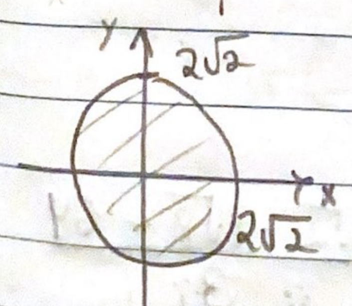
intersec: $(x^2+y^2+z^2=16,\ z=\sqrt{x^2+y^2})$

$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=16$$

$$x^2+y^2+x^2+y^2=16 \Rightarrow 2x^2+2y^2=16$$

$$x^2+y^2=8 \quad , \quad r=\sqrt{8}$$

$$r = 2\sqrt{2}$$

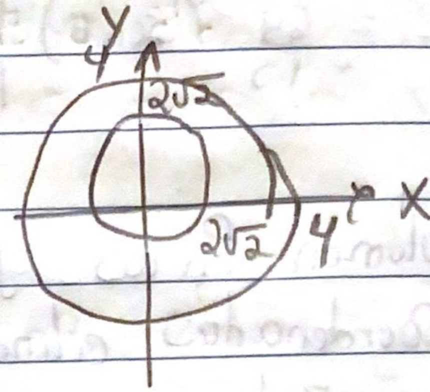


Parte externa: V2

$-\sqrt{16-r^2} \leq z \leq \sqrt{16-r^2}$

$2\sqrt{2} \leq r \leq 4$

$0 \leq \theta \leq 2\pi$



$$\underbrace{\int_0^{2\pi}\int_0^{2\sqrt{2}}\int_{-\sqrt{16-r^2}}^{r} r\,dz\,dr\,d\theta}_{V1} + \underbrace{\int_0^{2\pi}\int_{2\sqrt{2}}^{4}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta}_{V2}$$

V1 =

$$I = \int_{-\sqrt{16-r^2}}^{r} r\,dz = rz\Big|_{-\sqrt{16-r^2}}^{r} = \left(r+\sqrt{16-r^2}\right)r$$

$$\int_0^{2\sqrt{2}} r^2 + r\sqrt{16-r^2}\,dr$$

$u = \sqrt{16-r^2}$

$u^2 = 16-r^2$

$2u\,du = -2r\,dr$

$-u\,du = r\,dr$

$$\frac{r^3}{3}\Big|_0^{2\sqrt{2}} + \int_\rho^9 (-1)u\cdot u\,du$$

$$\left(\frac{(2\sqrt{2})^3}{3} - 0\right) + \int_\rho^9 -u^2\,du$$

$$+ -\frac{u^3}{3}\Big|_\rho^9$$

$$\frac{16\sqrt{2}}{3} + -\left[\frac{\left(\sqrt{16-r^2}\right)^3}{3}\right]_0^{2\sqrt{2}}$$

$$\frac{16\sqrt{2}}{3} + -\left[\frac{8\sqrt{8}}{3} - \frac{64}{3}\right] = \frac{16\sqrt{2} - 8\sqrt{8} + 64}{3}$$

$$\int_0^{2\pi} \frac{16\sqrt{2}-8\sqrt{8}+64}{3}\, d\theta = \frac{16\sqrt{2}-8\sqrt{8}+64}{3}\left[\theta\right]_0^{2\pi}$$

$$= \frac{2\pi\,(16\sqrt{2}-8\sqrt{8}+64)}{3}$$

$V_2 =$

$$I = \int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\, dz = rz \Big|_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} = r\left(\sqrt{16-r^2}\right) + r\left(\sqrt{16-r^2}\right)$$

$$= 2r\sqrt{16-r^2}$$

$$\int_{2\sqrt{2}}^{4} 2r\sqrt{16-r^2}\, dr$$

$$\left.\begin{aligned} u &= \sqrt{16-r^2} \\ u^2 &= 16-r^2 \\ 2u\,du &= -2r\,dr \\ -2u\,du &= 2r\,dr \end{aligned}\right\} \quad \int_{\rho}^{9} u\cdot\left((-1)2u\,du\right) = -2\int_{\rho}^{9} u^2\, du$$

$$= -2\left[\frac{u^3}{3}\right]_{\rho}^{9} = \frac{-2}{3}\left[\left(\sqrt{16-r^2}\right)^3\right]_{2\sqrt{2}}^{4}$$

$$= -2/3\left[0 - 8\sqrt{8}\right] = \frac{16\sqrt{8}}{3}$$

$$\int_0^{2\pi} 16\sqrt{8}/3\, d\theta = 16\sqrt{8}/3\left[\theta\right]_0^{2\pi} = 2\pi\left(\frac{16\sqrt{8}}{3}\right)$$

$$V_1 + V_2 = \frac{2\pi}{3}\left(16\sqrt{2}-8\sqrt{8}+64\right) + \frac{2\pi}{3}\left(16\sqrt{8}\right)$$

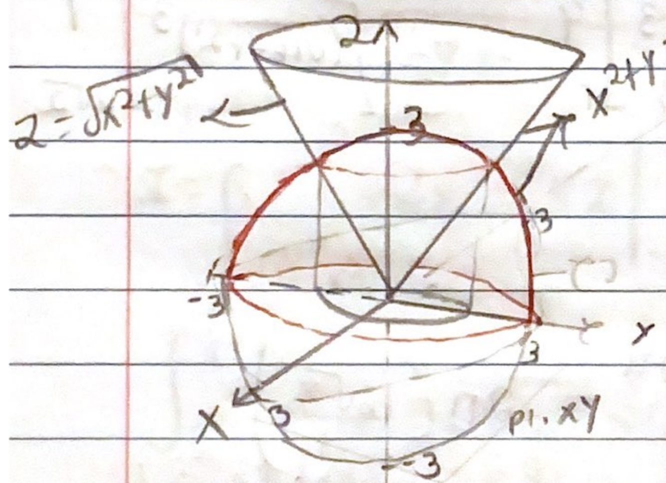
$$= \frac{128\pi}{3} + \frac{64\pi\sqrt{2}}{3}$$

$$= \frac{64}{3}\pi\left(2+\sqrt{2}\right)$$

Volumen: En los ejercicios 33 a 36, utilizar coordenadas esféricas para calcular el volumen del sólido.

33) Sólido interior $x^2+y^2+z^2=9$, exterior $z=\sqrt{x^2+y^2}$ y arriba del plano xy.

$$V=\iiint_E dV$$



R: (esférica)

$0 \le \rho \le 3$

$\pi/4 \le \phi \le \pi/2$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^3 \rho^2 \operatorname{sen}\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi} d\theta \cdot \int_{\pi/4}^{\pi/2} \operatorname{sen}\phi \, d\theta \cdot \int_0^3 \rho^2 \, d\rho$$

$$\theta \Big|_0^{2\pi} \cdot -\cos\phi \Big|_{\pi/4}^{\pi/2} \cdot \frac{\rho^3}{3}\Big|_0^3$$

$$(2\pi)\left(-\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{4}\right)\right)\cdot\left(\frac{3^3}{3}\right)$$

$$2\pi\cdot\left(0+\frac{\sqrt{2}}{2}\right)\cdot(9) = \frac{18\sqrt{2}\,\pi}{2} = 9\sqrt{2}\,\pi \text{ unid}^3$$