[MAA 1.2-1.3] ARITHMETIC SEQUENCES

SOLUTIONS

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O. Practice questions

SEQUENCES (IN GENERAL)

1. (a) $u_1 = 2$, $S_1 = 2$.

(b) $u_1 = 5$, $S_2 = 7$.

(c) $S_5 = 2 + 5 + 10 + 3 + 7 = 27$.

2. (a) $u_2 = 2$, $u_{20} = 20$.

(b) $S_5 = 1 + 2 + 3 + 4 + 5 = 15$.

(c) $u_n = n$.

3. (a) 10, 20, 30

(b) $u_{10} = 100$

(c) $S_1 = 10$, $S_2 = 30$, $S_3 = 60$

4. (a) 10, 40, 90

(b) $u_{10} = 1000$

(c) $S_1 = 10$, $S_2 = 50$, $S_3 = 140$

5. (a) 10, 100, 1000

(b) $u_{10} = 10{,}000{,}000{,}000$

(c) $S_1 = 10$, $S_2 = 110$, $S_3 = 1110$

6. (a) 10, 20, 30

(b) $S_1 = 10$, $S_2 = 30$, $S_3 = 60$

7. (a) 10, 30, 70

(b) $S_1 = 10$, $S_2 = 40$, $S_3 = 110$

8. (a) $\sum_{r=1}^{3}(2r) = 2 + 4 + 6 = 12$

(b) $\sum_{r=1}^{3} r^2 = 1 + 4 + 9 = 14$

(c) $\sum_{r=1}^{3} 2^r = 2 + 4 + 8 = 14$

9. (a) $A = \sum_{r=1}^{10}(2r^2 + 1) = 780$, $B = \sum_{r=1}^{20}(2r^2 + 1) = 5760$,

$C = \sum_{r=11}^{20}(2r^2 + 1) = 4980$

(b) $A + C = B$ or $C = B - A$

ARITHMETIC SEQUENCES

10. $u_1 = 11$, $d = 4$.

(a) $u_{101} = 11 + 100 \times 4 = 411$, $S_{101} = \frac{101}{2}(11 + 411) =$ 21311

(b) $S_{20} = \frac{20}{2}(2 \times 11 + 19 \times 4) = 980$

(c) $u_n = 11 + (n-1)4 = 11 + 4n - 4 = 4n + 7$

(d) $u_n = 51 \Leftrightarrow 4n + 7 = 51 \Leftrightarrow 4n = 44 \Leftrightarrow n = 11$

11. (a) d=8 (b) 75 (c) 385 (d) $n = 21$

12. (a) $u_1 = 10$ $d = 5$ (b) $u_n = 5n+5$ (c) $S_n = \frac{n}{2}(5n + 15)$ (d) $u_n = 105$ and $S_{20} = 1150$

13. (a) $u_1 = 10$ $u_2 = 15$ (b) $d = 5$ (c) $S_3 = 45$ $S_4 = 70$ (d) $S_n = \frac{n}{2}(5n + 15)$

14. (a) (i) 15 (ii) 40400 (iii) 40385 (b) 30200

15. (a) 23 (b) 99 (c) 1265

A. Exam style questions (SHORT)

16. (a) $u_1 = 7, d = 2.5$

$u_{41} = u_1 + (n-1)d = 7 + 40 \times 2.5 = 107$

(b) $S_{101} = \frac{101}{2}[2(7) + 100 \times 2.5] = \frac{101(264)}{2} = 13332$

17. $S_5 = \frac{5}{2}\{2 + 32\} = 85$

OR

$a = 2, a + 4d = 32 \Rightarrow 4d = 30 \Rightarrow d = 7.5$

$S_5 = \frac{5}{2}(4 + 4(7.5)) = \frac{5}{2}(4 + 30) = 85$

18. (a) $u_3 = u_1 + 2d \Leftrightarrow 8 = 2 + 2d \Leftrightarrow d = 3$

(b) $u_{20} = 2 + 19 \times 3 = 59$

(c) $S_{20} = \frac{20}{2}(2 + 59) = 610$ **OR** $S_{20} = \frac{20}{2}(2 \times 2 + 19 \times 3) = 610$

19. $u_4 = 40 \Leftrightarrow 5 + 3d = 40 \Leftrightarrow d = \frac{35}{3}$

$u_2 = 5 + \frac{35}{3} = \frac{50}{3}$

20. Arithmetic sequence

$u_1 = 200$ $d = 30$

(a) Distance in final week = 200 + 51 × 30= 1730 m

(b) Total distance = $\frac{52}{2}$ [2×200 + 51×.30] = 50180m

21. (a) Arithmetic sequence
$u_1 = 15 \quad d = 2 \quad n = 20$
$u_{20} = 15 + 19 \times 2 = 53$ (that is, 53 seats in the 20th row)

(b) $S_{20} = \frac{20}{2}(2 \times 15 + 19 \times 2)$ (or $\frac{20}{2}(15 + 53)$)
$= 680$ (that is, 680 seats in total)

22. (a) $u_1 = 1000$, $u_n = 1000 + (n - 1)250 = 10\,000$
$n = 37$.
She runs 10 km on the 37th day.

(b) $S_{37} = \frac{37}{2}(1000 + 10\,000) = 203\,500$. She has run a total of 203.5 km

23. (a) $T_1 = 100 \quad d = 25$
$T_{17} = 100 + 16 \times 25 = \$500$

(b) $S_{17} = \frac{17}{2}(100 + 500) = \$5100$

24. (a) $45000 + 4 \times 1750 = 52000$ USD

(b) $\frac{10}{2}(2(45000) + 9 \times (1750)) = 528750$ USD (Accept 529000)

25. (a) $20 = u_1 + 3d$

$32 = u_1 + 7d$

$d = 3$ (and $u_1 = 11$)

(b) $\frac{10}{2}(2 \times 11 + 9 \times 3) = 245$

26. (a) $u_{21} = 24 + 20 \times (16) = 344$

(b) $S_{31} = \frac{31}{2}[2(24) + (30)(16)] = 8184$

27. (a) $u_{20} = u_1 + 19d \Leftrightarrow 64 = 7 + 19d,\ d = \frac{64 - 7}{19} = 3$

(b) $u_n = 3709 \Leftrightarrow 3709 = 7 + 3(n - 1) \Leftrightarrow n = 1235$

28. $u_2 = u_1 + d = 7$ and $S_4 = \frac{4}{2}(2u_1 + 3d) = 12$

$\Rightarrow \begin{matrix} u_1 + d = 7 \\ 4u_1 + 6d = 12 \end{matrix} \quad \Rightarrow u_1 = 15,\ d = -8$

29. $u_2 = u_1 + d \Rightarrow u_1 + d = 7$

$S_5 = \frac{5}{2}(2u_1 + 4d) = 50 \Rightarrow 2u_1 + 4d = 20$

$d = 3$

30. (a) $u_{27} = 263 \Leftrightarrow 263 = u_1 + 26 \times 11 \Leftrightarrow u_1 = -23$

(b) (i) $516 = -23 + (n - 1) \times 11 \Leftrightarrow n = 50$

(ii) $S_{50} = \frac{50(-23 + 516)}{2} = 12325$ **OR** $S_{50} = \frac{50(2 \times (-23) + 49 \times 11)}{2} = 12325$

31. (a) $d = 3$

$u_{101} = 2 + 100 \times 3 = 302$

(b) $152 = 2 + (n - 1) \times 3 \Leftrightarrow 150 = (n - 1) \times 3 \Leftrightarrow 50 = n - 1 \Leftrightarrow n = 51$

32. (a) $S_n = \frac{n}{2}(2 \times 2 + 3(n - 1)) = \frac{n}{2}(3n + 1)$

(b) $\frac{n}{2}(3n + 1) = 1365 \Rightarrow 3n^2 + n - 2730 = 0$

$\Rightarrow n = 30$ or $n = \frac{-91}{3}$, hence $n = 30$

33. (a) $u_4 = u_1 + 3d \Leftrightarrow 16 = -2 + 3d \Leftrightarrow d = 6$

(b) $u_n = u_1 + (n - 1)\,6 \Leftrightarrow 11998 = -2 + (n - 1)6 \Leftrightarrow n = \frac{11998 + 2}{6} + 1 = 2001$

34. (a) (i) $-37 = u_1 + 20d$

$-3 = u_1 + 3d$

$-34 = 17d \Leftrightarrow d = -2$

(ii) $-3 = u_1 - 6 \Rightarrow u_1 = 3$

(b) $u_{10} = 3 + 9 \times -2 = -15$

$S_{10} = \frac{10}{2}(3 + (-15)) = -60$

35. (a) $\frac{20}{2}\{2(-7) + 19d\} = 620 \Leftrightarrow d = 4$

(b) $u_{78} = -7 + 77(4) = 301$

36. (a) 3, 6, 9

(b) (i) sum of an AP $\sum_{n=1}^{20} 3n = \frac{20}{2}\left(2 \times 3 + (20 - 1) \times 3\right) = 630$

(ii) **METHOD 1**

$\sum_{n=1}^{100} 3n = \frac{100}{2}(2 \times 3 + 99 \times 3) = 15150$

$\sum_{n=21}^{100} 3n = 15150 - 630 = 14520$

METHOD 2

first term is 63, the number of terms is 80

$\sum_{n=21}^{100} 3n = \frac{80}{2}(63 + 300) = 14520$

37. (a) $u_1 = 1, u_2 = -1, u_3 = -3$

(b) $S_{20} = \frac{20}{2}(2 \times 1 + 19 \times -2)\ (= 10(2 - 38)) = -360$

38. (a) $d = 2$

(b) (i) $5 + 2n = 115 \Rightarrow n = 55$

(ii) $u_1 = 7$

$$S_{55} = \frac{55}{2}(7+115) = 3355 \textbf{ OR } S_{55} = \frac{55}{2}(2(7)+54(2)), = 3355 \textbf{ OR } \sum_{k=1}^{55}(5+2k) = 3355$$

39. (a) common difference is 6

(b) $u_n = 1353 \Leftrightarrow 1353 = 3 + (n-1)6 \Leftrightarrow n = 226$

(c) $S_{226} = \dfrac{226(3+1353)}{2}$ **OR** $S_{226} = \dfrac{226}{2}(2 \times 3 + 225 \times 6)$

$S_{226} = 153\ 228$ (153 000 is also accepted)

40. **METHOD 1**

substituting into formula for S_{40} : $1900 = \dfrac{40(u_1 + 106)}{2} \Leftrightarrow u_1 = -11$

substituting into formula for u_{40} : $106 = -11 + 39d \Leftrightarrow d = 3$

METHOD 2

substituting into formulas for S_{40} and u_{40}

$20(2u_1 + 39d) = 1900$

$u_1 + 39d = 106$

Solution: $u_1 = -11$, $d = 3$

41. (a) $u_{20} = u_1 + 19d \Leftrightarrow 64 = 7 + 19d, \Leftrightarrow d = 3$

(b) $3709 = 7 + 3(n-1) \Leftrightarrow 3709 = 3n + 4 \Leftrightarrow n = 1235$

42. (a) n=21 (b) 1575

43. (a) $u_1 = 5$ and $d = 8$

$u_n = u_1 + (n-1)d \Rightarrow u_n = 8n - 3$

(b) $8n - 3 < 400 \Rightarrow 8n < 403$

$n < 50.375$

Therefore, there are 50 terms less than 400.

44. (a) $u_1 = S_1 = 7$

(b) $u_2 = S_2 - u_1 = 18 - 7 = 11$

$d = 11 - 7 = 4$

(c) $u_4 = u_1 + (n-1)d = 7 + 3(4)$

$u_4 = 19$

45. Arithmetic sequence $d = 3$

$n = 1250$

$$S = \frac{1250}{2}(3 + 3750) = 2\ 345\ 625 \textbf{ OR } S = \frac{1250}{2}(6 + 1249 \times 3) = 2\ 345\ 625$$

46. $81 = \dfrac{n}{2}(1.5 + 7.5) \Rightarrow n = 18$

$1.5 + 17d = 7.5 \Rightarrow d = \dfrac{6}{17}$

47. $17 + (n-1)10 = 417 \Leftrightarrow 10(n-1) = 400$ so $n = 41$

$S_{41} = \frac{41}{2}(2(17) + 40(10)) = 41(17 + 200) = 8897$ **OR** $S_{41} = \frac{41}{2}(17 + 417) = \frac{41}{2}(434) = 8897$

48. (a) $n = 51$, $S_{51} = 4182$

(b) $\sum_{r=1}^{51}(3r+4) = 4182$ (it is same sum!)

49. **(a)** $u_n = 5 + 4(n-1) = 4n + 1$

(b) $\sum_{r=1}^{21}(4r+1)$

50. Arithmetic progression: 85, 78, 71, ...

$u_1 = 85, d = -7$

$u_n = 85 - 7(n-1) = 92 - 7n$ $u_n > 0 \Rightarrow n \le 13.$

$S_{13} = \frac{13}{2}(u_1 + u_{13}) = \frac{13}{2}(85 + 1) = 559$

51. 4^{th} term $= a + 3d$

8^{th} term $= a + 7d$

20^{th} term $= a + 19d$

We have two relations:

$a + 7d = 2(a + 3d) \Leftrightarrow a + 7d = 2a + 6d \Leftrightarrow a = d$

$a + 19d = 4000$

The solution is $d = 200$ (and $a = 200$)

52. $u_1 = -6$ and $d = 7$

$S_n = \frac{n}{2}(2u_1 + (n-1)d) \Rightarrow S_n = \frac{n}{2}(7n - 19)$

Solving $S_n > 10000 \Rightarrow n > 54.8$, The least number of terms is 55

53. (a) $(2a + 4) - (a + 3) = (a + 9) - (2a + 4) \Leftrightarrow 2a = 4 \Leftrightarrow a = 2$

(b) the terms are 5, 9, 11 which are indeed in AS with $d = 2$

54. $(2a + b + 7) - (a - b) = (a - b) - 2$

$(a - 3b) - (2a + b + 7) = (2a + b + 7) - (a - b)$

$0 = 2a + 6b + 14$

$b = -3$ $a = 2$

55. Let a be the first term and d be the common difference of the arithmetic sequence.

Then $\dfrac{a+4d}{a+11d} = \dfrac{6}{13}$

So $13a + 52d = 6a + 66d \Rightarrow 7a = 14d \Rightarrow a = 2d$.

Since each term is positive, both a and d are positive.

We are given $a(a + 2d) = 32$, setting $a = 2d$, we get $2d(2d + 2d) = 8d^2 = 32$.

$\Rightarrow d = \pm 2$.

Hence, $d = 2$ and $a = 4$ and sum to 100 terms of this sequence is

$\dfrac{100}{2}\{(2)(4) + (100 - 1)2\}. = 10\,300$

56. (a) $S_1=1$, $S_2=8$, hence $u_1=1$, $u_2=7$

(b) $u_n = S_n - S_{n-1} = [3n^2 - 2n] - [3(n-1)^2 - 2(n-1)] = 6n - 5$

OR

$u_n = 1+6(n-1) = 6n - 5$

57. (a) $S_n = 2n^2 - n$

$n=1 \Rightarrow S_1 = u_1 = 2-1 = 1$

$n=2 \Rightarrow S_2 = u_1 + u_2 = 8-2 = 6 \Rightarrow u_2 = 5$

$n=3 \Rightarrow S_3 = u_1 + u_2 + u_3 = 18-3 = 15 \Rightarrow u_3 = 9$

(b) $u_n = S_n - S_{n-1}$

$\Rightarrow u_n = 2n^2 - n - \left(2(n-1)^2 - (n-1)\right)$

$\Rightarrow u_n = 2n^2 - n - \left(2n^2 - 4n + 2 - n + 1\right)$

$\Rightarrow u_n = 4n - 3$

58. (a) $S_4 = 68$ $S_5 = 105$ $u_5 = 37$ (b) $u_n = 8n - 3$ (c) $u_n - u_{n-1} = 8$

B. Exam style questions (LONG)

59. (a) 24 (b) 97 (c) 1224 (d) 21

60. (a) $u_1 = 1$, $n = 20$, $u_{20} = 20$ ($u_1 = 1$, $n = 20$, $d = 1$)

$S_{20} = \dfrac{(1+20)20}{2}$ $\left(\text{or } S = \dfrac{20}{2}(2\times 1 + 19\times 1)\right)$ $= 210$

(b) Let there be n cans in bottom row

$S_n = 3240 \Leftrightarrow \dfrac{(1+n)n}{2} = 3240 \Leftrightarrow n^2 + n - 6480 = 0 \Leftrightarrow n = 80$ or $n = -81$

So $n = 80$

(c) (i) $S = \dfrac{(1+n)n}{2} \Leftrightarrow 2S = n^2 + n \Leftrightarrow n^2 + n - 2S = 0$

(ii) **METHOD 1**

Substituting $S = 2100$: $n^2 + n - 4200 = 0 \Leftrightarrow n = 64.3$, $n = -65.3$

n must be a (positive) integer, this equation does not have integer solutions.

METHOD 2

Trial and error: $S_{64} = 2080$, $S_{65} = 2145$ integer not possible here

61. (a) $u_1 = 1$, $d = 3$ $u_{11} = 31$

(b) $S_n = \frac{n}{2}(2 + (n-1) \times 3) = \frac{n}{2}(2 + 3n - 3) = \frac{n}{2}(3n - 1).$

(c) (i) $\frac{100}{2}(3 \times 100 - 1) = 14950$

(d) (i) $\frac{n}{2}(3n - 1) = 477 \Leftrightarrow 3n^2 - n = 954 \Leftrightarrow 3n^2 - n - 954 = 0$

(ii) 18

62. (a) $(5k - 2) - (2k + 3) = (10k - 15) - (5k - 2)$

$\Leftrightarrow 5k - 2 - 2k - 3 = 10k - 15 - 5k + 2$

$\Leftrightarrow 3k - 5 = 5k - 13 \quad \Leftrightarrow -2k = -8$ or $2k = 8$

$\Leftrightarrow k = 4$

(b) 11, 18, 25

(c) 7

(d) $U_{20} = 11 + 19 \times 7 = 144$

(e) $S_{15} = \frac{15}{2}(2 \times 11 + 14 \times 7) = 900$

63. (a) (i) 6, 9, 12   (ii) 3

(b) $x^2 - (x^2 - 3) = 4x - x^2 \Leftrightarrow x^2 - 4x^2 + 3 = 0 \Leftrightarrow x = 1$ or $x = 3$

So the other value is 1.

(d) (i) -2, 1, 4

(ii) 3

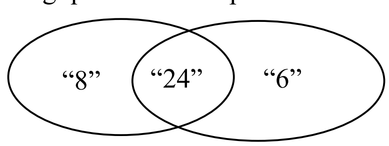
(iii) -2 + 1 + 4 + 7 = 10

64. The multiples of 8 form an arithmetic sequence and have the form $8n$

(a) $\frac{900}{8} = 112.5$, hence $\sum_{n=1}^{112} 8n = 50624$

(b) $\frac{100}{8} = 12.5$ hence $\sum_{x=13}^{112} 8n = 50000$

For the remaining questions it helps to draw a Venn diagram for the multiples of 8 and 6:



(c) The common multiples are the multiples of 24. $\frac{900}{24} = 37.5$, hence $\sum_{n=1}^{37} 24n = 16872$

(d) $\sum_{x=13}^{112} 8n - \sum_{x=1}^{37} 24n = 50624 - 16872 = 33752$

(e) $\frac{900}{6} = 150$ hence the sum of the multiples of 6 is $\sum_{n=1}^{149} 6n = 67050$

The sum of all multiples either of 8 or of 6 is

$\sum_{x=13}^{112} 8n + \sum_{n=1}^{149} 6n - \sum_{x=1}^{37} 24n = 50624 + 67050 - 16872 = 100802$

65. The sizes form an AS with $u_1 = 1$ and $d = 3$

(a) (i) the 20th group contains $u_{20} = 1 + 19 \times 3 = 58$ numbers

(ii) the last term of the 20th group is $S_{20} = 590$

(b) (i) the nth group contains $u_n = 1 + (n-1)3 = 3n - 2$ numbers

(ii) the last term of the nth group is $S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(1 + 3n - 2) = \frac{n(3n-1)}{2}$

(c) the first term of the 20th group is $S_{19} + 1 = 533$

(d) the sum of the terms in the 20th group is $\frac{58}{2}(533 + 590) = 32567$