1.3 Lines and Planes

We are all familiar with the equation of a line in the Cartesian plane. We now want to consider lines in $\mathbb{R}^2$ from a vector point of view. The insights we obtain from this approach will allow us to generalize to lines in $\mathbb{R}^3$ and then to planes in $\mathbb{R}^3$. Much of the linear algebra we will consider in later chapters has its origins in the simple geometry of lines and planes; the ability to visualize these and to think geometrically about a problem will serve you well.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

In the *xy*-plane, the general form of the equation of a line is $ax + by = c$. If $b \neq 0$, then the equation can be rewritten as $y = -(a/b)x + c/b$, which has the form $y = mx + k$. [This is the slope-intercept form; m is the slope of the line, and the point with coordinates $(0, k)$ is its y-intercept.] To get vectors into the picture, let's consider an example.

Example 1.26

The line ℓ with equation $2x + y = 0$ is shown in Figure 1.53. It is a line with slope -2 passing through the origin. The left-hand side of the equation is in the form of a dot product; in fact, if we let $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then the equation becomes $\mathbf{n} \cdot \mathbf{x} = 0$. The vector $\mathbf{n}$ is perpendicular to the line—that is, it is *orthogonal* to any vector $\mathbf{x}$ that is parallel to the line (Figure 1.54)—and it is called a ***normal vector*** to the line. The equation $\mathbf{n} \cdot \mathbf{x} = 0$ is the *normal form* of the equation of ℓ.

Another way to think about this line is to imagine a particle moving along the line. Suppose the particle is initially at the origin at time $t = 0$ and it moves along the line in such a way that its x-coordinate changes 1 unit per second. Then at $t = 1$ the particle is at $(1, -2)$, at $t = 1.5$ it is at $(1.5, -3)$, and, if we allow negative values of t (that is, we consider where the particle was in the past), at $t = -2$ it is (or was) at

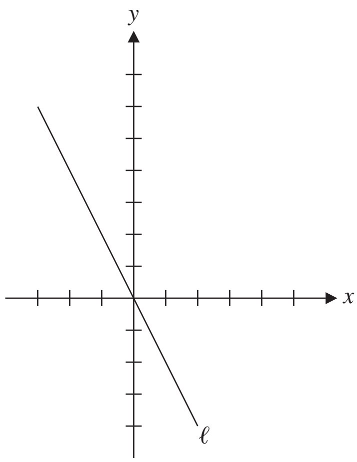

Figure 1.53
The line $2x + y = 0$

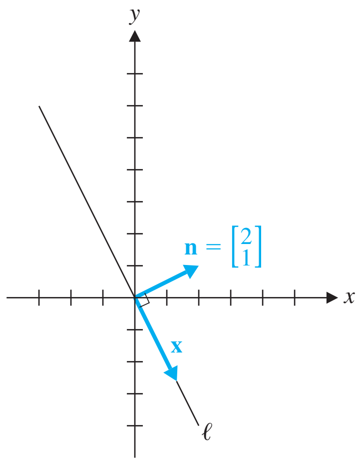

Figure 1.54
A normal vector $\mathbf{n}$

The Latin word *norma* refers to a carpenter's square, used for drawing right angles. Thus, a *normal* vector is one that is perpendicular to something else, usually a plane.

$(-2, 4)$. This movement is illustrated in Figure 1.55. In general, if $x = t$, then $y = -2t$, and we may write this relationship in vector form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ -2t \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

What is the significance of the vector $\mathbf{d} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$? It is a particular vector parallel to ℓ, called a ***direction vector*** for the line. As shown in Figure 1.56, we may write the equation of ℓ as $\mathbf{x} = t\mathbf{d}$. This is the *vector form* of the equation of the line.

If the line does not pass through the origin, then we must modify things slightly.

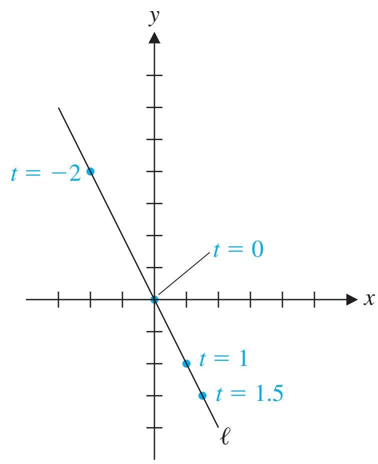

Figure 1.55

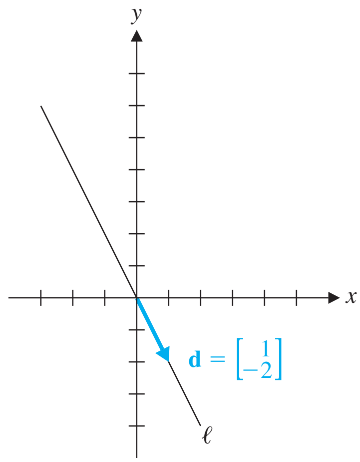

Figure 1.56
A direction vector **d**

Example 1.27

Consider the line ℓ with equation $2x + y = 5$ (Figure 1.57). This is just the line from Example 1.26 shifted upward 5 units. It also has slope -2, but its y-intercept is the point $(0, 5)$. It is clear that the vectors **d** and **n** from Example 1.26 are, respectively, a direction vector and a normal vector for this line too.

Thus, **n** is orthogonal to every vector that is parallel to ℓ. The point $P = (1, 3)$ is on ℓ. If $X = (x, y)$ represents a general point on ℓ, then the vector $\overrightarrow{PX} = \mathbf{x} - \mathbf{p}$ is parallel to ℓ and $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ (see Figure 1.58). Simplified, we have $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$. As a check, we compute

$$\mathbf{n} \cdot \mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 2x + y \quad \text{and} \quad \mathbf{n} \cdot \mathbf{p} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 5$$

Thus, the normal form $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ is just a different representation of the general form of the equation of the line. (Note that in Example 1.26, **p** was the zero vector, so $\mathbf{n} \cdot \mathbf{p} = 0$ gave the right-hand side of the equation.)

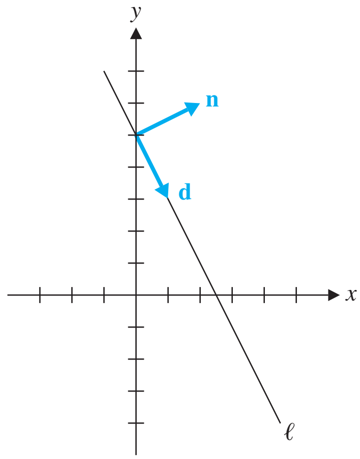

Figure 1.57
The line $2x + y = 5$

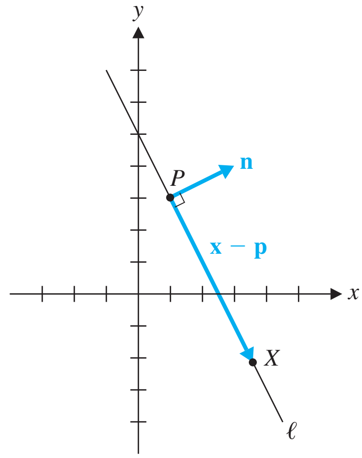

Figure 1.58
$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

These results lead to the following definition.

Definition The ***normal form of the equation of a line*** ℓ in $\mathbb{R}^2$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for ℓ.

The ***general form of the equation of*** ℓ is $ax + by = c$, where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$ is a normal vector for ℓ.

Continuing with Example 1.27, let us now find the vector form of the equation of ℓ. Note that, for each choice of $\mathbf{x}$, $\mathbf{x} - \mathbf{p}$ must be parallel to—and thus a multiple of—the direction vector $\mathbf{d}$. That is, $\mathbf{x} - \mathbf{p} = t\mathbf{d}$ or $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ for some scalar t. In terms of components, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad (1)$$

or

$$\begin{aligned} x &= 1 + t \\ y &= 3 - 2t \end{aligned} \qquad (2)$$

Equation (1) is the vector form of the equation of ℓ, and the componentwise equations (2) are called *parametric equations* of the line. The variable t is called a *parameter*.

How does all of this generalize to $\mathbb{R}^3$? Observe that the vector and parametric forms of the equations of a line carry over perfectly. The notion of the slope of a line in $\mathbb{R}^2$—which is difficult to generalize to three dimensions—is replaced by the more convenient notion of a direction vector, leading to the following definition.

The word *parameter* and the corresponding adjective *parametric* come from the Greek words *para*, meaning "alongside," and *metron*, meaning "measure." Mathematically speaking, a parameter is a variable in terms of which other variables are expressed—a new "measure" placed alongside old ones.

Definition The ***vector form of the equation of a line*** ℓ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + t\mathbf{d}$$

where $\mathbf{p}$ is a specific point on ℓ and $\mathbf{d} \neq \mathbf{0}$ is a direction vector for ℓ.

The equations corresponding to the components of the vector form of the equation are called ***parametric equations*** of ℓ.

We will often abbreviate this terminology slightly, referring simply to the general, normal, vector, and parametric equations of a line or plane.

Example 1.28 Find vector and parametric equations of the line in $\mathbb{R}^3$ through the point $P = (1, 2, -1)$, parallel to the vector $\mathbf{d} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$.

Solution The vector equation $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

The parametric form is

$$\begin{aligned} x &= 1 + 5t \\ y &= 2 - t \\ z &= -1 + 3t \end{aligned}$$

Remarks

- The vector and parametric forms of the equation of a given line ℓ are not unique—in fact, there are infinitely many, since we may use any point on ℓ to determine $\mathbf{p}$ and any direction vector for ℓ. However, all direction vectors are clearly multiples of each other.

In Example 1.28, (6, 1, 2) is another point on the line (take $t = 1$), and $\begin{bmatrix} 10 \\ -2 \\ 6 \end{bmatrix}$ is another direction vector. Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix} + s\begin{bmatrix} 10 \\ -2 \\ 6 \end{bmatrix}$$

gives a different (but equivalent) vector equation for the line. The relationship between the two parameters s and t can be found by comparing the parametric equations: For a given point (x, y, z) on ℓ, we have

$$\begin{aligned} x &= 1 + 5t = 6 + 10s \\ y &= 2 - t = 1 - 2s \\ z &= -1 + 3t = 2 + 6s \end{aligned}$$

implying that

$$\begin{aligned} -10s + 5t &= 5 \\ 2s - t &= -1 \\ -6s + 3t &= 3 \end{aligned}$$

Each of these equations reduces to $t = 1 + 2s$.

- Intuitively, we know that a line is a *one-dimensional* object. The idea of "dimension" will be clarified in Chapters 3 and 6, but for the moment observe that this idea appears to agree with the fact that the vector form of the equation of a line requires *one* parameter.

Example 1.29

One often hears the expression "two points determine a line." Find a vector equation of the line ℓ in $\mathbb{R}^3$ determined by the points $P = (-1, 5, 0)$ and $Q = (2, 1, 1)$.

Solution We may choose any point on ℓ for $\mathbf{p}$, so we will use P (Q would also be fine).

A convenient direction vector is $\mathbf{d} = \overrightarrow{PQ} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ (or any scalar multiple of this). Thus, we obtain

$$\begin{aligned} \mathbf{x} &= \mathbf{p} + t\mathbf{d} \\ &= \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} \end{aligned}$$

Planes in $\mathbb{R}^3$

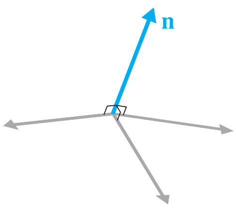

Figure 1.59
n is orthogonal to infinitely many vectors

The next question we should ask ourselves is, How does the general form of the equation of a line generalize to $\mathbb{R}^3$? We might reasonably guess that if $ax + by = c$ is the general form of the equation of a line in $\mathbb{R}^2$, then $ax + by + cz = d$ might represent a line in $\mathbb{R}^3$. In normal form, this equation would be $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$, where $\mathbf{n}$ is a normal vector to the line and $\mathbf{p}$ corresponds to a point on the line.

To see if this is a reasonable hypothesis, let's think about the special case of the equation $ax + by + cz = 0$. In normal form, it becomes $\mathbf{n} \cdot \mathbf{x} = 0$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. However, the set of all vectors $\mathbf{x}$ that satisfy this equation is the set of all vectors orthogonal to $\mathbf{n}$. As shown in Figure 1.59, vectors in infinitely many directions have this property, determining a family of parallel *planes*. So our guess was incorrect: It appears that $ax + by + cz = d$ is the equation of a plane—not a line—in $\mathbb{R}^3$.

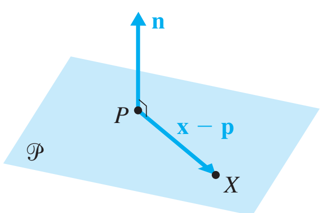

Figure 1.60
$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$

Let's make this finding more precise. Every plane $\mathcal{P}$ in $\mathbb{R}^3$ can be determined by specifying a point $\mathbf{p}$ on $\mathcal{P}$ and a nonzero vector $\mathbf{n}$ normal to $\mathcal{P}$ (Figure 1.60). Thus, if $\mathbf{x}$ represents an arbitrary point on $\mathcal{P}$, we have $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$ or $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$. If $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then, in terms of components, the equation becomes $ax + by + cz = d$ (where $d = \mathbf{n} \cdot \mathbf{p}$).

Definition The ***normal form of the equation of a plane*** $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \quad \text{or} \quad \mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$$

where $\mathbf{p}$ is a specific point on $\mathcal{P}$ and $\mathbf{n} \neq \mathbf{0}$ is a normal vector for $\mathcal{P}$.

The ***general form of the equation of*** $\mathcal{P}$ is $ax + by + cz = d$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a normal vector for $\mathcal{P}$.

Note that any scalar multiple of a normal vector for a plane is another normal vector.

Example 1.30

Find the normal and general forms of the equation of the plane that contains the point $P = (6, 0, 1)$ and has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution With $\mathbf{p} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have $\mathbf{n} \cdot \mathbf{p} = 1 \cdot 6 + 2 \cdot 0 + 3 \cdot 1 = 9$, so the normal equation $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ becomes the general equation $x + 2y + 3z = 9$.

Geometrically, it is clear that parallel planes have the same normal vector(s). Thus, their general equations have left-hand sides that are multiples of each other. So, for example, $2x + 4y + 6z = 10$ is the general equation of a plane that is parallel to the plane in Example 1.30, since we may rewrite the equation as $x + 2y + 3z = 5$—from which we see that the two planes have the same normal vector $\mathbf{n}$. (Note that the planes do not coincide, since the right-hand sides of their equations are distinct.)

We may also express the equation of a plane in vector or parametric form. To do so, we observe that a plane can also be determined by specifying one of its points P (by the vector $\mathbf{p}$) and *two* direction vectors $\mathbf{u}$ and $\mathbf{v}$ parallel to the plane (but not parallel to each other). As Figure 1.61 shows, given any point X in the plane (located

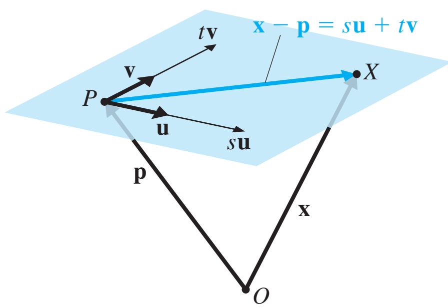

Figure 1.61
$\mathbf{x} - \mathbf{p} = s\mathbf{u} + t\mathbf{v}$

by $\mathbf{x}$), we can always find appropriate multiples $s\mathbf{u}$ and $t\mathbf{v}$ of the direction vectors such that $\mathbf{x} - \mathbf{p} = s\mathbf{u} + t\mathbf{v}$ or $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$. If we write this equation componentwise, we obtain parametric equations for the plane.

Definition The ***vector form of the equation of a plane*** $\mathcal{P}$ in $\mathbb{R}^3$ is

$$\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$$

where $\mathbf{p}$ is a point on $\mathcal{P}$ and $\mathbf{u}$ and $\mathbf{v}$ are direction vectors for $\mathcal{P}$ ($\mathbf{u}$ and $\mathbf{v}$ are nonzero and parallel to $\mathcal{P}$, but not parallel to each other).

The equations corresponding to the components of the vector form of the equation are called ***parametric equations*** of $\mathcal{P}$.

Example 1.31

Find vector and parametric equations for the plane in Example 1.30.

Solution We need to find two direction vectors. We have one point $P = (6, 0, 1)$ in the plane; if we can find two other points Q and R in $\mathcal{P}$, then the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ can serve as direction vectors (unless by bad luck they happen to be parallel!). By trial and error, we observe that $Q = (9, 0, 0)$ and $R = (3, 3, 0)$ both satisfy the general equation $x + 2y + 3z = 9$ and so lie in the plane. Then we compute

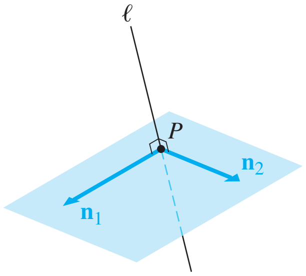

Figure 1.62
Two normals determine a line

$$\mathbf{u} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$$

which, since they are not scalar multiples of each other, will serve as direction vectors. Therefore, we have the vector equation of $\mathcal{P}$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$$

and the corresponding parametric equations,

$$\begin{aligned} x &= 6 + 3s - 3t \\ y &= 3t \\ z &= 1 - s - t \end{aligned}$$

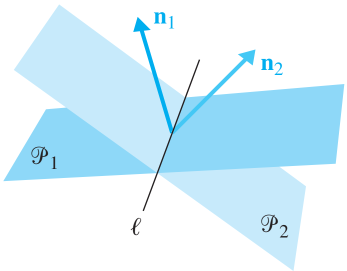

Figure 1.63
The intersection of two planes is a line

[What would have happened had we chosen $R = (0, 0, 3)$?]

Remarks

- A plane is a two-dimensional object, and its equation, in vector or parametric form, requires *two* parameters.
- As Figure 1.59 shows, given a point P and a nonzero vector $\mathbf{n}$ in $\mathbb{R}^3$, there are infinitely many lines through P with $\mathbf{n}$ as a normal vector. However, P and two nonparallel normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ do serve to locate a line ℓ uniquely, since ℓ must then be the line through P that is perpendicular to the plane with equation $\mathbf{x} = \mathbf{p} + s\mathbf{n}_1 + t\mathbf{n}_2$ (Figure 1.62). Thus, a line in $\mathbb{R}^3$ can also be specified by a pair of equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \end{aligned}$$

one corresponding to each normal vector. But since these equations correspond to a pair of nonparallel planes (why nonparallel?), this is just the description of a line as the intersection of two nonparallel planes (Figure 1.63). Algebraically, the line consists of all points (x, y, z) that simultaneously satisfy both equations. We will explore this concept further in Chapter 2 when we discuss the solution of systems of linear equations.

Tables 1.2 and 1.3 summarize the information presented so far about the equations of lines and planes.

Observe once again that a single (general) equation describes a line in $\mathbb{R}^2$ but a plane in $\mathbb{R}^3$. [In higher dimensions, an object (line, plane, etc.) determined by a single equation of this type is usually called a ***hyperplane.***] The relationship among the

Table 1.2 Equations of Lines in $\mathbb{R}^2$

Normal Form	General Form	Vector Form	Parametric Form
$\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$	$ax+by=c$	$\mathbf{x}=\mathbf{p}+t\mathbf{d}$	$\begin{cases}x=p_1+td_1\\y=p_2+td_2\end{cases}$

Table 1.3 Lines and Planes in $\mathbb{R}^3$

	Normal Form	General Form	Vector Form	Parametric Form
Lines	$\begin{cases}\mathbf{n}_1\cdot\mathbf{x}=\mathbf{n}_1\cdot\mathbf{p}_1\\\mathbf{n}_2\cdot\mathbf{x}=\mathbf{n}_2\cdot\mathbf{p}_2\end{cases}$	$\begin{cases}a_1x+b_1y+c_1z=d_1\\a_2x+b_2y+c_2z=d_2\end{cases}$	$\mathbf{x}=\mathbf{p}+t\mathbf{d}$	$\begin{cases}x=p_1+td_1\\y=p_2+td_2\\z=p_3+td_3\end{cases}$
Planes	$\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$	$ax+by+cz=d$	$\mathbf{x}=\mathbf{p}+s\mathbf{u}+t\mathbf{v}$	$\begin{cases}x=p_1+su_1+tv_1\\y=p_2+su_2+tv_2\\z=p_3+su_3+tv_3\end{cases}$

dimension of the object, the number of equations required, and the dimension of the space is given by the "balancing formula":

(*dimension* of the object) + (*number* of general equations) = dimension of the space

The higher the dimension of the object, the fewer equations it needs. For example, a plane in $\mathbb{R}^3$ is two-dimensional, requires one general equation, and lives in a three-dimensional space: $2 + 1 = 3$. A line in $\mathbb{R}^3$ is one-dimensional and so needs $3 - 1 = 2$ equations. Note that the dimension of the object also agrees with the number of parameters in its vector or parametric form. Notions of "dimension" will be clarified in Chapters 3 and 6, but for the time being, these intuitive observations will serve us well.

We can now find the distance from a point to a line or a plane by combining the results of Section 1.2 with the results from this section.

Example 1.32

Find the distance from the point $B = (1, 0, 2)$ to the line ℓ through the point $A = (3, 1, 1)$ with direction vector $\mathbf{d} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$.

Solution As we have already determined, we need to calculate the length of $\overrightarrow{PB}$, where P is the point on ℓ at the foot of the perpendicular from B. If we label $\mathbf{v} = \overrightarrow{AB}$, then $\overrightarrow{AP} = \text{proj}_{\mathbf{d}}(\mathbf{v})$ and $\overrightarrow{PB} = \mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})$ (see Figure 1.64). We do the necessary calculations in several steps.

Step 1: $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix}1\\0\\2\end{bmatrix} - \begin{bmatrix}3\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-1\\1\end{bmatrix}$

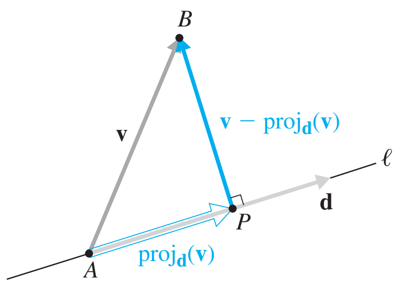

Figure 1.64
$d(B, \ell) = \|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\|$

Step 2: The projection of **v** onto **d** is

$$\begin{aligned} \text{proj}_{\mathbf{d}}(\mathbf{v}) &= \left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} \\ &= \left(\frac{(-1)\cdot(-2) + 1\cdot(-1) + 0\cdot 1}{(-1)^2 + 1 + 0}\right)\begin{bmatrix}-1\\1\\0\end{bmatrix} \\ &= \tfrac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} \\ &= \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} \end{aligned}$$

Step 3: The vector we want is

$$\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v}) = \begin{bmatrix}-2\\-1\\1\end{bmatrix} - \begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} = \begin{bmatrix}-\frac{3}{2}\\ -\frac{3}{2}\\ 1\end{bmatrix}$$

Step 4: The distance $d(B, \ell)$ from B to ℓ is

$$\|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| = \left\|\begin{bmatrix}-\frac{3}{2}\\ -\frac{3}{2}\\ 1\end{bmatrix}\right\|$$

Using Theorem 1.3(b) to simplify the calculation, we have

$$\begin{aligned} \|\mathbf{v} - \text{proj}_{\mathbf{d}}(\mathbf{v})\| &= \tfrac{1}{2}\left\|\begin{bmatrix}-3\\-3\\2\end{bmatrix}\right\| \\ &= \tfrac{1}{2}\sqrt{9 + 9 + 4} \\ &= \tfrac{1}{2}\sqrt{22} \end{aligned}$$

Note

- In terms of our earlier notation, $d(B, \ell) = d(\mathbf{v}, \text{proj}_{\mathbf{d}}(\mathbf{v}))$.

In the case where the line ℓ is in $\mathbb{R}^2$ and its equation has the general form $ax + by = c$, the distance $d(B, \ell)$ from $B = (x_0, y_0)$ is given by the formula

$$d(B, \ell) = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}} \tag{3}$$

You are invited to prove this formula in Exercise 39.

Example 1.33

Find the distance from the point $B = (1, 0, 2)$ to the plane $\mathcal{P}$ whose general equation is $x + y - z = 1$.

Solution In this case, we need to calculate the length of $\overrightarrow{PB}$, where P is the point on $\mathcal{P}$ at the foot of the perpendicular from B. As Figure 1.65 shows, if A is any point on $\mathcal{P}$ and we situate the normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ of $\mathcal{P}$ so that its tail is at A, then we need to find the length of the projection of $\overrightarrow{AB}$ onto $\mathbf{n}$. Again we do the necessary calculations in steps.

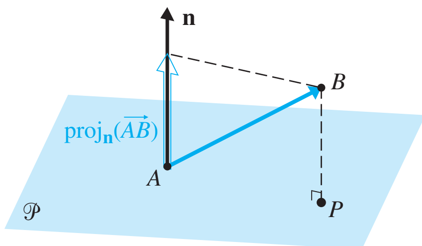

Figure 1.65

$d(B, \mathcal{P}) = \|\text{proj}_\mathbf{n}(\overrightarrow{AB})\|$

Step 1: By trial and error, we find any point whose coordinates satisfy the equation $x + y - z = 1$. $A = (1, 0, 0)$ will do.

Step 2: Set

$$\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Step 3: The projection of $\mathbf{v}$ onto $\mathbf{n}$ is

$$\begin{aligned} \text{proj}_\mathbf{n}(\mathbf{v}) &= \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n} \\ &= \left(\frac{1 \cdot 0 + 1 \cdot 0 - 1 \cdot 2}{1 + 1 + (-1)^2}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ &= -\frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \end{aligned}$$

Step 4: The distance $d(B, \mathscr{P})$ from B to $\mathscr{P}$ is

$$\begin{aligned}\|\text{proj}_{\mathbf{n}}(\mathbf{v})\| &= \left|-\tfrac{2}{3}\right| \left\|\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\| \\ &= \tfrac{2}{3}\left\|\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\| \\ &= \tfrac{2}{3}\sqrt{3}\end{aligned}$$

In general, the distance $d(B, \mathscr{P})$ from the point $B = (x_0, y_0, z_0)$ to the plane whose general equation is $ax + by + cz = d$ is given by the formula

$$d(B, \mathscr{P}) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} \tag{4}$$

You will be asked to derive this formula in Exercise 40.

Exercises 1.3

In Exercises 1 and 2, write the equation of the line passing through P with normal vector **n** *in (a) normal form and (b) general form.*

1. $P = (0, 0), \mathbf{n} = \begin{bmatrix}3\\2\end{bmatrix}$

2. $P = (2, 1), \mathbf{n} = \begin{bmatrix}3\\-4\end{bmatrix}$

In Exercises 3–6, write the equation of the line passing through P with direction vector **d** *in (a) vector form and (b) parametric form.*

3. $P = (1, 0), \mathbf{d} = \begin{bmatrix}-1\\3\end{bmatrix}$

4. $P = (3, -3), \mathbf{d} = \begin{bmatrix}-1\\1\end{bmatrix}$

5. $P = (0, 0, 0), \mathbf{d} = \begin{bmatrix}1\\-1\\4\end{bmatrix}$

6. $P = (-3, 1, 2), \mathbf{d} = \begin{bmatrix}1\\0\\5\end{bmatrix}$

In Exercises 7 and 8, write the equation of the plane passing through P with normal vector **n** *in (a) normal form and (b) general form.*

7. $P = (0, 1, 0), \mathbf{n} = \begin{bmatrix}3\\2\\1\end{bmatrix}$

8. $P = (-3, 1, 2), \mathbf{n} = \begin{bmatrix}1\\0\\5\end{bmatrix}$

In Exercises 9 and 10, write the equation of the plane passing through P with direction vectors **u** *and* **v** *in (a) vector form and (b) parametric form.*

9. $P = (0, 0, 0), \mathbf{u} = \begin{bmatrix}2\\1\\2\end{bmatrix}, \mathbf{v} = \begin{bmatrix}-3\\2\\1\end{bmatrix}$

10. $P = (4, -1, 3), \mathbf{u} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{v} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$

In Exercises 11 and 12, give the vector equation of the line passing through P and Q.

11. $P = (1, -2), Q = (3, 0)$

12. $P = (4, -1, 3), Q = (2, 1, 3)$

In Exercises 13 and 14, give the vector equation of the plane passing through P, Q, and R.

13. $P = (1, 1, 1), Q = (4, 0, 2), R = (0, 1, -1)$

14. $P = (1, 0, 0), Q = (0, 1, 0), R = (0, 0, 1)$

15. Find parametric equations and an equation in vector form for the lines in $\mathbb{R}^2$ with the following equations:

(a) $y = 3x - 1$ **(b)** $3x + 2y = 5$

16. Consider the vector equation $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$, where $\mathbf{p}$ and $\mathbf{q}$ correspond to distinct points P and Q in $\mathbb{R}^2$ or $\mathbb{R}^3$.

(a) Show that this equation describes the line segment $\overline{PQ}$ as t varies from 0 to 1.
(b) For which value of t is $\mathbf{x}$ the midpoint of $\overline{PQ}$, and what is $\mathbf{x}$ in this case?
(c) Find the midpoint of $\overline{PQ}$ when $P = (2, -3)$ and $Q = (0, 1)$.
(d) Find the midpoint of $\overline{PQ}$ when $P = (1, 0, 1)$ and $Q = (4, 1, -2)$.
(e) Find the two points that divide $\overline{PQ}$ in part (c) into three equal parts.
(f) Find the two points that divide $\overline{PQ}$ in part (d) into three equal parts.

17. Suggest a "vector proof" of the fact that, in $\mathbb{R}^2$, two lines with slopes m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

18. The line ℓ passes through the point $P = (1, -1, 1)$ and has direction vector $\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. For each of the following planes $\mathcal{P}$, determine whether ℓ and $\mathcal{P}$ are parallel, perpendicular, or neither:

(a) $2x + 3y - z = 1$ (b) $4x - y + 5z = 0$
(c) $x - y - z = 3$ (d) $4x + 6y - 2z = 0$

19. The plane $\mathcal{P}_1$ has the equation $4x - y + 5z = 2$. For each of the planes $\mathcal{P}$ in Exercise 18, determine whether $\mathcal{P}_1$ and $\mathcal{P}$ are parallel, perpendicular, or neither.

20. Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is perpendicular to the line with general equation $2x - 3y = 1$.

21. Find the vector form of the equation of the line in $\mathbb{R}^2$ that passes through $P = (2, -1)$ and is parallel to the line with general equation $2x - 3y = 1$.

22. Find the vector form of the equation of the line in $\mathbb{R}^3$ that passes through $P = (-1, 0, 3)$ and is perpendicular to the plane with general equation $x - 3y + 2z = 5$.

23. Find the vector form of the equation of the line in $\mathbb{R}^3$ that passes through $P = (-1, 0, 3)$ and is parallel to the line with parametric equations

$$\begin{aligned} x &= 1 - t \\ y &= 2 + 3t \\ z &= -2 - t \end{aligned}$$

24. Find the normal form of the equation of the plane that passes through $P = (0, -2, 5)$ and is parallel to the plane with general equation $6x - y + 2z = 3$.

25. A cube has vertices at the eight points (x, y, z), where each of x, y, and z is either 0 or 1. (See Figure 1.34.)

(a) Find the general equations of the planes that determine the six faces (sides) of the cube.
(b) Find the general equation of the plane that contains the diagonal from the origin to $(1, 1, 1)$ and is perpendicular to the xy-plane.
(c) Find the general equation of the plane that contains the side diagonals referred to in Example 1.22.

26. Find the equation of the set of all points that are equidistant from the points $P = (1, 0, -2)$ and $Q = (5, 2, 4)$.

In Exercises 27 and 28, find the distance from the point Q to the line ℓ.

27. $Q = (2, 2)$, ℓ with equation $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

28. $Q = (0, 1, 0)$, ℓ with equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

In Exercises 29 and 30, find the distance from the point Q to the plane $\mathcal{P}$.

29. $Q = (2, 2, 2)$, $\mathcal{P}$ with equation $x + y - z = 0$

30. $Q = (0, 0, 0)$, $\mathcal{P}$ with equation $x - 2y + 2z = 1$

Figure 1.66 suggests a way to use vectors to locate the point R on ℓ that is closest to Q.

31. Find the point R on ℓ that is closest to Q in Exercise 27.

32. Find the point R on ℓ that is closest to Q in Exercise 28.

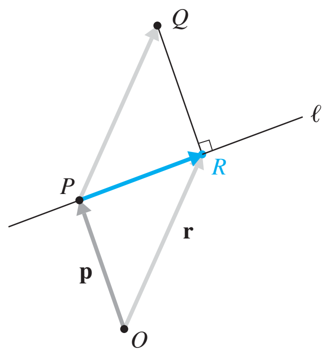

Figure 1.66
$\mathbf{r} = \mathbf{p} + \overrightarrow{PR}$

Figure 1.67 suggests a way to use vectors to locate the point R on $\mathcal{P}$ that is closest to Q.

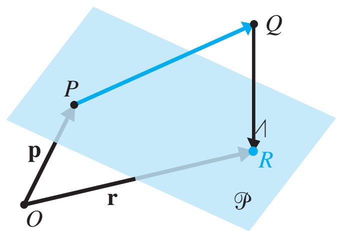

Figure 1.67

$\mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{QR}$

33. Find the point R on $\mathcal{P}$ that is closest to Q in Exercise 29.

34. Find the point R on $\mathcal{P}$ that is closest to Q in Exercise 30.

In Exercises 35 and 36, find the distance between the parallel lines.

35. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + t\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

36. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 37 and 38, find the distance between the parallel planes.

37. $2x + y - 2z = 0$ and $2x + y - 2z = 5$

38. $x + y + z = 1$ and $x + y + z = 3$

39. Prove equation (3) on page 43.

40. Prove equation (4) on page 44.

41. Prove that, in $\mathbb{R}^2$, the distance between parallel lines with equations $\mathbf{n}\cdot\mathbf{x} = c_1$ and $\mathbf{n}\cdot\mathbf{x} = c_2$ is given by $\dfrac{|c_1 - c_2|}{\|\mathbf{n}\|}$.

42. Prove that the distance between parallel planes with equations $\mathbf{n}\cdot\mathbf{x} = d_1$ and $\mathbf{n}\cdot\mathbf{x} = d_2$ is given by $\dfrac{|d_1 - d_2|}{\|\mathbf{n}\|}$.

If two nonparallel planes $\mathcal{P}_1$ and $\mathcal{P}_2$ have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ and θ is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, then we define the angle between $\mathcal{P}_1$ and $\mathcal{P}_2$ to be either θ or $180° - \theta$, whichever is an acute angle. (Figure 1.68)

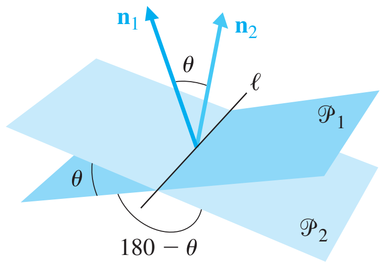

Figure 1.68

In Exercises 43–44, find the acute angle between the planes with the given equations.

43. $x + y + z = 0$ and $2x + y - 2z = 0$

44. $3x - y + 2z = 5$ and $x + 4y - z = 2$

In Exercises 45–46, show that the plane and line with the given equations intersect, and then find the acute angle of intersection between them.

45. The plane given by $x + y + 2z = 0$ and the line given by
$$x = 2 + t$$
$$y = 1 - 2t$$
$$z = 3 + t$$

46. The plane given by $4x - y - z = 6$ and the line given by
$$x = t$$
$$y = 1 + 2t$$
$$z = 2 + 3t$$

Exercises 47–48 explore one approach to the problem of finding the projection of a vector onto a plane. As Figure 1.69 shows, if $\mathcal{P}$ is a plane through the origin in $\mathbb{R}^3$ with normal vector $\mathbf{n}$, and $\mathbf{v}$ is a vector in $\mathbb{R}^3$, then $\mathbf{p} = \text{proj}_{\mathcal{P}}(\mathbf{v})$ is a vector in $\mathcal{P}$ such that $\mathbf{v} - c\mathbf{n} = \mathbf{p}$ for some scalar c.

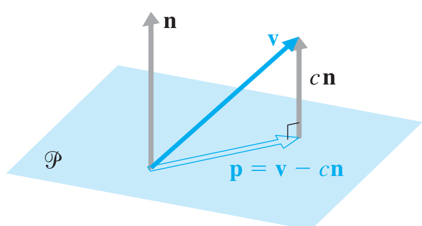

Figure 1.69
Projection onto a plane

47. Using the fact that $\mathbf{n}$ is orthogonal to every vector in $\mathcal{P}$ (and hence to $\mathbf{p}$), solve for c and thereby find an expression for $\mathbf{p}$ in terms of $\mathbf{v}$ and $\mathbf{n}$.

48. Use the method of Exercise 43 to find the projection of

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

onto the planes with the following equations:

(a) $x + y + z = 0$ **(b)** $3x - y + z = 0$

(c) $x - 2z = 0$ **(d)** $2x - 3y + z = 0$