Some Analysis of Nonlinear Pendulum Motion **WCEvans 9/23**

Pendulum motion, with its displacement variable *theta*, an angle measured positive counterclockwise from the vertical, and zero when the pendulum is hanging straight down, is described by a nonlinear second-order ordinary differential equation (ODE), the time-dependent solution to which involves Jacobi elliptic functions. These are the inverses of certain elliptic integrals that arise in solving the ODE. In the accompanying Geogebra simulation that ODE solution is found using numerical methods.

The true period *T* (as opposed to the linearized, small-angle approximation often used) of the motion is [1, p179] [2, p313] [3, p193]:

$$\text{T = 4 * Sqrt[L / g] * Elliptic} \tag{1}$$

where

$$\text{Elliptic = Integrate[1 / Sqrt[1 - } k^2 \text{ * } Sin^2(x) \text{], \{ x, 0, } \pi/2 \text{ \}]} \tag{2}$$

and *L* is the pendulum length, and *g* is the gravitational acceleration. Note that, for the square root to be real-valued, and thus the motion to be periodic, k^2 must be less than unity. Also, if *k* is zero, the integral is just π/2, making the period *T = 2π*Sqrt[L/g]*, which is the period of the linearized, small-angle motion. This essential parameter *k* is defined as [4, p53]:

$$k^2 = \text{Total Energy / Max Potential Energy}$$

or

$$k^2 = E_{total} \text{ / (mg* 2L)}$$

The maximum potential energy is available when the pendulum is at a displacement angle of π radians (180 degrees), or straight up at the top of its swing, which makes its gravitational height *2L* (zero is at the bottom point of the swing). The total mechanical energy is of course the sum of the kinetic and potential energies:

$$E_{total} = \tfrac{1}{2} mL^2 \text{ * thetadot0}^2 + \text{mgL(1 - Cos[theta0])} \tag{3}$$

where *thetadot0* is the initial (time=0) angular velocity imparted to the pendulum, if any, and *theta0* is the initial angular displacement, if any. The first term in eq(3) is just the usual kinetic energy $\tfrac{1}{2}mv^2$ on an angular-motion basis. Notice that *L(1 - Cos[theta])* is the height of the pendulum above the zero level. Then we have

$$k^2 = E_{total} \text{ / (2mgL) = L/(4g) * thetadot0}^2 + \tfrac{1}{2} \text{ (1 - Cos[theta0])} \tag{4}$$

This gives us the ability to calculate the true period *T*, using the result of eq(4) in eq(2) and that result in eq(1). This is based on both the initial displacement and initial angular velocity; the latter is rarely included in such calculations. The simulation verifies that this calculated period is correct, by noting the observed period on the time-dependent motion plot (easiest to see when using a zero *theta0*).

We can also find the amplitude of the motion, i.e., the maximum angular displacement *thetamax*, using conservation of energy (CoE), and recognizing that when the pendulum is at *thetamax* its velocity will be zero. CoE says that the total mechanical energy at any time *t* is equal to the energy input to the system at time zero, which is the energy that initiated the pendulum's motion. Thus,

$$\tfrac{1}{2} mL^2 \text{ * thetadot(t)}^2 + \text{mgL(1 - Cos[theta(t)]) } = \tfrac{1}{2} mL^2 \text{ * thetadot0}^2 + \text{mgL(1 - Cos[theta0])}$$

and the first term will be zero when *t* is such that *theta(t) = thetamax*. Then

$$\text{mgL(1 - Cos[thetamax]) } = \tfrac{1}{2} mL^2 \text{ * thetadot0}^2 + \text{mgL(1 - Cos[theta0])}$$

from which

$$\text{thetamax = ArcCos[Cos[theta0] - L/(2g) * thetadot0}^2 \text{]} \tag{5}$$

and again we have included both the initial displacement and initial angular velocity. With a bit of algebra we can also write this as

$$\text{thetamax = ArcCos[1 - } 2k^2 \text{] = ArcCos[1 - } E_{total} \text{ / mgL]} \tag{6}$$

thus relating the amplitude of the motion to the total energy of the system. Note that the energy must be "normalized" in some manner in order to remove its dependence on the mass of the pendulum. In this format we can easily see that the total energy must not exceed the maximum potential energy *2mgL*, or the inverse cosine will be undefined, and the motion is no longer periodic (it goes around). The eq(6) variation of the amplitude of the motion with the energy parameter k^2 is shown in the right panel of the figure below.

An interesting case of the pendulum motion is when it approaches a displacement angle of π radians, or straight up. If it has just a bit more energy, it will pass π radians and go completely around. Then the motion is no longer periodic, in the formal sense, although it does repeat. If we set *thetamax* to π and solve eq(5) for the initial angular velocity "*thetadot0pi*" that will just attain that amplitude,

$$\text{thetadot0pi = Sqrt[2g/L (1 + Cos[theta0])]} \tag{7}$$

This function has its maximum when *theta0* = 0, and it decreases to zero when *theta0* = π. The latter case also leads to the parameter *k* being unity, which could lead to the elliptic integral denominator being zero and the integral is then undefined.

If we input a given amount of energy, the motion of the pendulum will be the same whether that energy input was potential (by only displacing the pendulum through an initial angle) or kinetic (by imparting an angular velocity only, with zero initial displacement). So, equating the terms in eq(3),

$$\tfrac{1}{2}\, mL^2 \ * \ \text{thetadot0}^2 \ = \ mgL(\, 1 \ - \ \text{Cos[theta0]}\,)$$

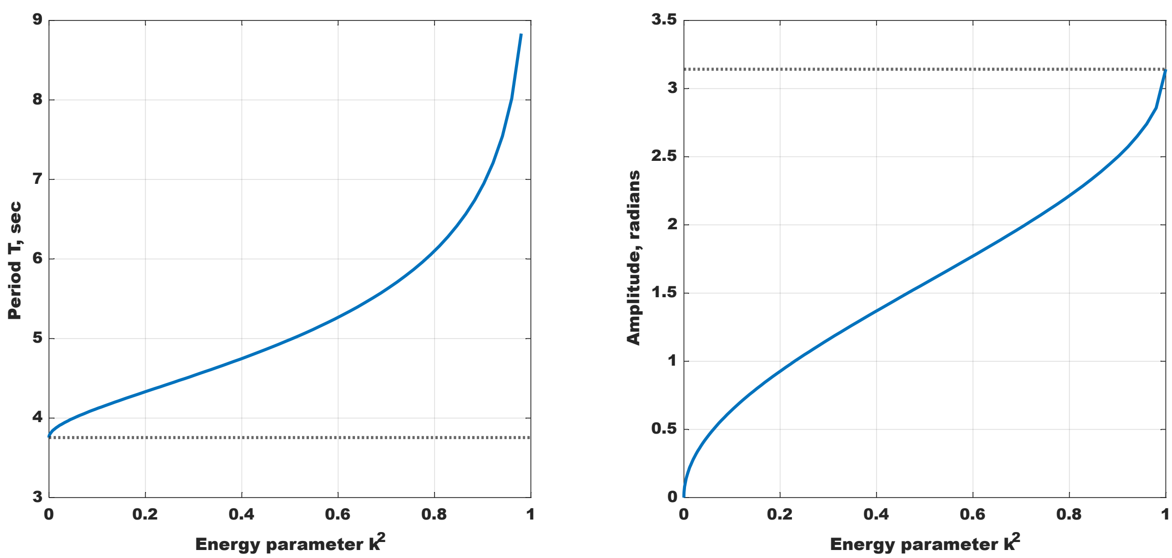
and solving for the *theta0* that corresponds (by equal energy input) to a given *thetadot0*, we find that

$$\text{theta0} \ = \ \text{ArcCos[}\ 1 \ - \ L/(2g) \ * \ \text{thetadot0}^2 \ \text{]} \qquad (8)$$

Be clear that each of these energies is isolated; potential only or kinetic only, not a mixture. We could solve for the kinetic input corresponding to a given potential input, but this is not generally useful. Having put in a certain amount of kinetic energy and seeing the system response, it is a natural question to ask what initial displacement *theta0* would have caused that same motion.

Consider the argument of the inverse cosine in eq(8); since this argument must be in the range -1 to +1, the largest *thetadot0* can be is *2Sqrt[g/L]*, just as is the case in eq(7). This corresponds to the amount of energy input that will cause the pendulum to go around in a circle, in this instance kinetic energy by itself. Of course, in general if the total energy input, potential plus kinetic, is such that the eq(4) parameter k^2 is greater than unity, the pendulum will go around and the motion is no longer periodic. In that case the variable *thetamax* is undefined, since the displacement angle *theta* will increase without limit.

We can see the relation of the period T of a 3.5 meter pendulum to the energy parameter k^2 in the left panel of the figure below. The horizontal line is the small-angle-motion period, which is also the T value given by eq(1) for zero k^2. Interestingly, this seems to imply a finite period when no energy is input to the system and there would therefore be no motion. The right panel shows the 3.5 meter pendulum amplitude as it varies with k^2. This shows a zero amplitude at zero energy, as we would expect.



References

(These have mathematically-detailed discussions of pendulum motion, near the pages cited.)

[1] O'Neill, *Advanced Engineering Mathematics*, 3rd Ed., Wadsworth, 1991.

[2] Ritger and Rose, *Differential Equations with Applications*, Dover reprint.

[3] Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover reprint.

[4] Corben and Stehle, *Classical Mechanics*, 2nd Ed. Dover reprint.