NOTES ON MOMENTUM, IMPULSE AND COLLISIONS

INTRODUCTION

In the *Principia*, Newton never mentions "momentum" but the phrase "quantity of motion" he used has been interpreted in more modern times to mean momentum. He wanted us to consider, simultaneously, both the mass and velocity of a moving object. Today we define momentum to be

$$\mathbf{p} \equiv m\,\mathbf{v} \tag{1}$$

where **p** is the momentum vector, and the velocity **v** is of course a vector as well. The units of momentum are N-s or kg-m/s. Newton actually stated in his second law that the "alteration of motion is ever proportional to the motive force impressed." In modern terms this amounts to saying that the time rate of change of momentum is equal to the force,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\,\mathbf{v})}{dt} = m\,\frac{d\mathbf{v}}{dt} = m\,\mathbf{a} \tag{2}$$

The familiar form of Newton's Second Law (**F** = *m***a**), which appears nowhere in the *Principia*, implicitly assumes that the mass is constant. This is usually the case, but it is not true for rockets, for example.

So, the "momentum", for our purposes, is just the product of the mass and velocity of an object. Since it is a vector quantity, it can be analyzed in 1D, 2D, or 3D. We will only work in 1D and 2D; of course 3D problems are much more tedious to analyze but involve no new physics.

IMPULSE

When a force is applied to an object, we should consider that this takes place over some time interval. Often we take that interval to be very short (like hitting a baseball), and over that short interval, the object does not move significantly. The “impulse” is defined to be

$$\mathbf{I} \equiv \int_0^{\tau} \mathbf{F}(t)\,dt = \Delta\mathbf{p} \tag{3}$$

This is an “integral of motion” as discussed during our study of work-energy. In that case we integrated the force over some finite displacement, to find the work done, or the change in the kinetic energy of the object upon which the force was acting. Here, we integrate the applied force over some finite time interval [0, τ] to get the change in the object’s momentum.

In general the actual time-dependent behavior of the “impulsive” force, *F(t)*, is not known, aside from the fact that it must start and end at zero, i.e., $F(0) = 0$ and $F(\tau) = 0$. However, we can say that, if the applied force is assumed to be constant at some *average* value over a short time interval, then the impulse applied to the object will be

$$\mathbf{I} = \overline{\mathbf{F}}\,\Delta t \tag{4}$$

The average value of the applied force could be found using

$$\overline{\mathbf{F}} = \frac{1}{\tau}\int_0^{\tau} F(t)\,dt \tag{5}$$

but this assumes we know the force *F(t)*, and in that case we could just use Eq(3). In problem statements, the average force is frequently given, so that we need Eq(5) to see how that average is related to the actual applied force.

Next we use Eq(2) to write, replacing the differentials with finite differences:

$$\mathbf{F} = m\,\mathbf{a} \quad \approx \quad m\,\frac{\Delta \mathbf{v}}{\Delta t}$$

from which we get the important relation

$$\overline{\mathbf{F}}\,\Delta t \;=\; \mathbf{I} \;=\; \Delta \mathbf{p} \;=\; m\,\Delta \mathbf{v} \;=\; m(\mathbf{v}_f - \mathbf{v}_i) \tag{6}$$

So, a given force applied over some (usually short) time interval will result in a change in the velocity of an object, assuming that the mass is constant, as it usually is. This change in velocity can of course involve a change in speed, or the direction of motion, or both.

CONSERVATION OF MOMENTUM

Conservation laws are among the most fundamental aspects of physics; we have already studied the conservation of energy. In the present case we have this important law:

If no net external force acts on a system of objects, then the total vector momentum of the system remains constant (i.e., is conserved).

The word "vector" is essential here, since the direction of motion of the objects in the system will vary in such a way that the net *vector* momentum of the system is conserved.

This conservation law (CoMo) is a direct consequence of Newton’s Third Law; alternatively, that Law is often considered to be a statement of CoMo. Since forces occur in pairs, and consist of the interaction of two objects, it stands to reason that if one force acts in one direction, and its reaction force acts in the opposite direction, then the *vector* impulses, and thus momentum changes, will add up to zero. These ideas can be expressed as

$$\mathbf{F}_{12} + \mathbf{F}_{21} = \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}\left(\mathbf{p}_1 + \mathbf{p}_2\right) = 0$$

If the time derivative of a quantity is zero, then the quantity is constant, so we see that the sum of the vector momenta of the system remains constant. The total momentum of the two-object system will not be changed by the application of an action-reaction force pair *within that system*. The law of conservation of momentum is a result of the fact that interactions within an isolated system are action-reaction pairs.

For example, consider the situation when firing an M-14 rifle, as Mr. Evans did, in the Army. Just before he pulls the trigger, neither the rifle nor the bullet is moving, so that the momentum of the system (rifle plus bullet) is zero. When the rifle is fired, the bullet attains a high velocity over a very short time interval, due to the application of a force within that system, namely, the expanding gases from the combustion of gunpowder.

That force is applied as long as the bullet is in the barrel of the rifle. The bullet then exits the barrel to the right, moving at a very high speed. The M-14, in order to conserve momentum (or we could also look at it as the reaction to the force applied to the bullet) moves abruptly to the left, into Mr. Evans's shoulder, causing him to exclaim "Dude! That hurts!"

We can write this conservation-law example mathematically as

$$m_{rifle}\,v_{rifle} \;+\; m_{bullet}\,v_{bullet} \;=\; 0$$

The rifle is of course much more massive than the bullet, so that its velocity is much smaller. But the rifle does move! How much (how fast) it moves can be estimated using

$$v_{rifle} = -\frac{m_{bullet}\, v_{bullet}}{m_{rifle}}$$

Note the negative sign, since the rifle moves in the opposite direction from the bullet.

COLLISIONS

One of the main applications of the law of conservation of momentum (CoMo) is in the study of collisions. A typical example of this is pool (billiards), where the ability to predict, in 2D, where the struck ball will go, and how fast it will go, is something that good players are able to do. (Without math.) Those collisions, however, are very complicated to analyze, due to the rolling motion of the ball, friction, and, especially, the spin used to control the motion after the collision. We will stick to more tractable situations.

We can write the CoMo mathematically as

$$\sum_{i=1}^{n} m_i\, \mathbf{v}_i = \mathbf{k}$$

where the sum is taken over all the objects that comprise the "system" we're analyzing; in most cases there are only two objects. The vector constant **k** is the momentum of the system; this is, often, zero. Since the impulsive (short-time) force that acts during the collision is internal to the two-object system, there can be no change in the system's total momentum due to the collision itself. Thus, *momentum is always conserved in collisions*.

To begin, we will work with just two objects, in one dimension, and we will treat a collision as "before" and "after" values. Then we can write the basic equation for collision analysis using CoMo:

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \qquad (7)$$

Here, v is the velocity before the collision and u is the velocity after the collision. The mass of course remains the same. Since we have six variables, the notation is important in keeping things orderly.

Since there are six variables in Eq(7) it is clear that we won't get very far in the analysis of collisions without some additional information. That information is found using the Conservation of Energy (CoE), which in this application is the conservation of kinetic energy (K):

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 \qquad (8)$$

In an **elastic collision**, kinetic energy is conserved, and we can use Eq(7) and Eq(8) to develop solutions. If kinetic energy is not conserved, we have an **inelastic collision**. One special case of the latter, that is easily treated, is where the two objects "stick together" after the collision, and then move together with the same velocity. Then we have, from Eq(7),

$$m_1 v_1 + m_2 v_2 = \left(m_1 + m_2\right) u_{12} \qquad (9)$$

where u_{12} is the after-collision velocity of the combined object. This can be solved in various ways, depending on the problem statement, and we can also find the K lost in the collision.

It can be shown without much difficulty that for an elastic collision Eq(7) and Eq(8) will lead to

$$v_1 + u_1 = v_2 + u_2 \qquad (10)$$

regardless of the masses of the two objects. Eq(10) is often, with a little rearranging, interpreted to say that the relative speed of the objects before the collision is the negative of the relative speed after.

With a bit of algebra it can be shown that the solutions for the after-collision velocities in an elastic collision are

$$\begin{aligned} u_1 &= v_1\left(\frac{m_1 - m_2}{m_1 + m_2}\right) + v_2\left(\frac{2\,m_2}{m_1 + m_2}\right) \\ u_2 &= v_1\left(\frac{2\,m_1}{m_1 + m_2}\right) + v_2\left(\frac{m_2 - m_1}{m_1 + m_2}\right) \end{aligned} \qquad (11)$$

Solutions for various special cases can be found using Eq(11). For example, if the masses are equal, we can immediately see that

$$u_1 = v_2 \qquad u_2 = v_1$$

so that the objects just exchange velocities. Other examples typically involve one mass being much larger than the other; for these problems, use limits.

COEFFICIENT OF RESTITUTION; COLLISION CALCULATIONS

Newton, in his experimental studies of collisions, developed a useful quantity called the coefficient of restitution. This is defined to be

$$\varepsilon \equiv \frac{\text{velocity of separation}}{\text{velocity of approach}} = -\frac{u_1 - u_2}{v_1 - v_2} \qquad (12)$$

The negative sign accounts for the change in direction. This coefficient varies (for our purposes) between zero and unity, corresponding to a perfectly inelastic, and perfectly elastic, collision, respectively. Eq(10) is a special case of Eq(12), with $\varepsilon = 1$.

If we have this parameter, it can be shown that the kinetic energy loss in any collision is given by

$$\Delta E_K = \frac{1}{2}\frac{m_1\,m_2}{m_1 + m_2}\left(v_1 - v_2\right)^2\left(\varepsilon^2 - 1\right) \qquad (13)$$

Note that Eq(13) finds this energy loss without explicitly solving for the after-collision velocities, and also observe that if the collision is perfectly elastic ($\varepsilon = 1$) then the E_K loss is zero.

The next issue is to write the general equations that we use to solve 1D or 2D collision problems, for two objects. If there are more objects, we extend these equations in the obvious manner. We can consider 1D collisions to be “direct” or “head-on” and 2D collisions are “oblique.” There is *no single solution that can be derived from these equations*; there are too many possible combinations of given and requested variables. We can make assumptions and develop a few simplified, special-case solutions, such as Eq(11), but this is the exception.

First, from CoMo we have the vector equation

$$m_1\,\mathbf{v}_1 + m_2\,\mathbf{v}_2 = m_1\,\mathbf{u}_1 + m_2\,\mathbf{u}_2$$

and expanding this in components along *x-y* axes that will be clear from the problem statement,

$$m_1\,v_1\cos\phi_1 + m_2\,v_2\cos\phi_2 = m_1\,u_1\cos\theta_1 + m_2\,u_2\cos\theta_2 \qquad (14)$$

$$m_1\,v_1\sin\phi_1 + m_2\,v_2\sin\phi_2 = m_1\,u_1\sin\theta_1 + m_2\,u_2\sin\theta_2 \qquad (15)$$

In more advanced courses, the initial approach angles φ and after-collision angles θ are measured with respect to the tangent to the surface of collision, and finding this can get complicated. For our purposes, the angles φ will be either zero, for motion of the object along the *x*-axis, which we can usually define in any convenient way, or 90 degrees, for motion along the *y*-axis. (An example of the latter situation is the cars-colliding-at-an-intersection problem.)

The after-collision angles θ are measured with respect to the *x*-axis, and *should be measured in a consistent way*, i. e., as positive counterclockwise from the *x*-axis. If you try to use negative angles, things will get confused; the signs of the trig functions will take care of themselves if you measure the angles in a consistent manner. Also remember that the magnitudes of the velocities *v* and *u* are, by definition, always positive. Making a sketch of the collision can help to ensure that the angles make sense.

The next equation uses the coefficient of restitution; in vector form this is

$$\left(\mathbf{u}_1 - \mathbf{u}_2\right) \bullet \mathbf{i} \; = \; -\,\varepsilon\left(\mathbf{v}_1 - \mathbf{v}_2\right) \bullet \mathbf{i}$$

where **i** is a unit vector that for our purposes is just the usual *x*-axis unit vector. Expanding this gives

$$\varepsilon\left(v_2 \cos\phi_2 \; - \; v_1 \cos\phi_1\right) \; = u_1 \cos\theta_1 \; - \; u_2 \cos\theta_1 \qquad (16)$$

The negative sign has been included on the RHS. Finally, we can use a version of the CoE relation that includes a term *Q* for energy lost in an inelastic collision:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \; = \; \frac{1}{2} m_1 u_1^2 \; + \; \frac{1}{2} m_2 u_2^2 \; + \; Q \qquad (17)$$

Eq(13) can be used to find a value for *Q*, as needed.

How to use Eq(14-17) to solve problems will depend, of course, upon what is given and what is to be found. There are many, many variations, some simple and some very complicated. There is no difficult physics in these problems (at this level), but the actual solutions can be, to say the least, tedious to develop, taking a lot of algebra and some trig. *Care in the use of notation is* ***essential*** *in these problems*.

A convenient, quick way to find the final velocities u_1 and u_2 for a 1D (direct) collision, if the masses and initial velocities are given, as they usually are, is as follows:

(1) Find the total (initial) momentum of the system:

$$A \equiv m_1 v_1 + \; m_2 v_2$$

(2) Find the difference in the initial velocities, multiplied by the coefficient of restitution:

$$B \equiv \varepsilon\left(v_2 - v_1\right)$$

(3) Define this matrix and use RREF:

$$\mathrm{RREF}\begin{pmatrix} m_1 & m_2 & A \\ 1 & -1 & B \end{pmatrix} \; = \; \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

In using either Eq(11) or the method above, be careful with the signs of the velocities. Note that for a perfectly inelastic collision ($\varepsilon = 0$), the two solutions for final velocity will be equal, since the two objects combine into a single object with one velocity.