[MAA 5.4] TANGENT AND NORMAL LINES

SOLUTIONS

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O. Practice questions

1. (a) $f(x)=2x^2-12x+10$, $f'(x)=4x-12$

(b) (i) At $x=1$, $y=0$, Point (1, 0)

$m_T=-8$, $m_T=\frac{1}{8}$

Tangent line: $y=-8(x-1)$ (i.e. $y=-8x+8$)

Normal line: $y=\frac{1}{8}(x-1)$ (i.e. $y=\frac{1}{8}x-\frac{1}{8}$)

(ii) At $x=2$, $y=-6$, Point (2, – 6)

$m_T=-4$, $m_T=\frac{1}{4}$

Tangent line: $y+6=-4(x-2)$ (**OR** $y=-4x+2$)

Normal line: $y+6=\frac{1}{4}(x-2)$ (**OR** $y=\frac{1}{4}x-\frac{13}{2}$)

(iii) At $x=3$, $y=-8$, Point (3, – 8)

$m_T=0$ (horizontal tangent)

Tangent line: $y=-8$, Normal line: $x=3$

2. $f(x)=2x^2-12x+10$, $f'(x)=4x-12$

(a) $4x-12=4 \Leftrightarrow 4x=16 \Leftrightarrow x=4$

Then $y=-6$ Point (4, – 6)

Tangent $y+6=4(x-4)$ (**OR** $y=4x-22$)

(b) $4x-12=-4 \Leftrightarrow 4x=8 \Leftrightarrow x=2$

Then $y=-6$ Point (2, – 6)

Tangent $y+6=-4(x-2)$ (**OR** $y=-4x+2$)

3. (a) $f'(x)=\mathrm{e}^x\times(-\sin x)+\cos x\times\mathrm{e}^x=\mathrm{e}^x\cos x-\mathrm{e}^x\sin x$

$m_T=f'(\pi)=\mathrm{e}^{\pi}\cos\pi-\mathrm{e}^{\pi}\sin\pi=-\mathrm{e}^{\pi}$

gradient of normal $m_N=\frac{1}{\mathrm{e}^{\pi}}$

(b) $m_T=f'\left(\frac{\pi}{4}\right)=0$

4. **METHOD A:** $f(x)=x^2$, tangent line $y=mx-25$

At the point of contact:

Equal functions: $f(x)=y \quad \Rightarrow \quad x^2=mx-25 \qquad (1)$

Equal derivatives: $f'(x)=y' \quad \Rightarrow \quad 2x=m \qquad (2)$

We solve the system:

(1) and (2): $x^2=(2x)x-25 \Leftrightarrow x^2=2x^2-25 \Leftrightarrow x^2=25 \Leftrightarrow x=\pm 5$

From (2): if $x=5$, $m=10$,

if $x=-5$, $m=-10$

METHOD B: $f(x)=x^2$, line $y=mx-25$

$$f(x)=y \Leftrightarrow x^2=mx-25 \Leftrightarrow x^2-mx+25=0$$

The line is a tangent if $\Delta=0 \Leftrightarrow m^2-100=0 \Leftrightarrow m=\pm 10$

5. (Only method A applies here!)

(a) $y=x^4$ and tangent line $y=mx-48$

At the point of contact:

Equal functions: $x^4=mx-48 \quad (1)$

Equal derivatives: $4x^3=m \quad (2)$

We solve the system:

(1) and (2): $x^4=(4x^3)x-48 \Leftrightarrow x^4=4x^4-48 \Leftrightarrow 3x^4=48 \Leftrightarrow x^4=16 \Leftrightarrow x=\pm 2$

From (2): If $x=2$, $m=32$, if $x=-2$, $m=-32$

(b) If $m=32$ the tangent line is $y=32x-48$

If $m=-32$ the tangent line is $y=-32x-48$

6. **METHOD A**

Line passing through A(0, - 48): $y+48=m(x-0) \Rightarrow y=mx-48$

[then we work as in question 5]

At the point of contact: $x^4=mx-48 \quad (1)$

$4x^3=m \quad (2)$

We solve the system:

(1) and (2): $x^4=(4x^3)x-48 \Leftrightarrow x^4=4x^4-48 \Leftrightarrow 3x^4=48 \Leftrightarrow x^4=16 \Leftrightarrow x=\pm 2$

From (2): If $x=2$, $m=32$, if $x=-2$, $m=-32$

If $m=32$ the tangent line is $y=32x-48$

If $m=-32$ the tangent line is $y=-32x-48$

METHOD B

We find the general tangent at (a,a^4)

$\frac{dy}{dx}=4x^3$, $m_T=4a^3$

Tangent line: $y-a^4=4a^3(x-a) \quad \rightarrow \quad y=4a^3x-3a^4$

Passes through A(0, - 48): $-3a^4=-48 \Leftrightarrow a^4=16 \Leftrightarrow a=\pm 2$

For $a=2$, $y=32x-48$

For $a=-2$, $y=-32x-48$

A. Exam style questions (SHORT)

7. Let $f(x) = 5x^2 + 10$, $f'(x) = 10x$

At P(1,15), $m_T = 10$, Tangent: $y - 15 = 10(x-1) \Rightarrow \; y = 10x + 5$,

8. $y = x^3 + 1 \qquad \frac{dy}{dx} = 3x^2$

At $x = 1$, $y = 2$, Point (1,2)

$\frac{dy}{dx} = 3x^{2,} \qquad m_T = 3$

Equation of tangent: $y - 2 = 3(x - 1) \Rightarrow y = 3x - 1$

$m_N = -\frac{1}{3}$

Equation of normal: $y - 2 = -\frac{1}{3}(x - 1) \Rightarrow 3y - 6 = -x + 1 \Rightarrow x + 3y - 7 = 0$ **OR** $y = -\frac{1}{3}x + 2\frac{1}{3}$

9. $f'(x) = 12x^2 + 2$

When $x = 1, f(1) = 6$

When $x = 1, f'(1) = 14$

Equation is $y - 6 = -\frac{1}{14}(x-1) \; \left(y = -\frac{1}{14}x + \frac{85}{14}, y = -0.0714x + 6.07\right)$

10. $y = (x-1)^4$, $\frac{dy}{dx} = 4(x-1)^3$

(i) At $x = 0$, $y = 1$, $m_T = -4$, $m_N = \frac{1}{4}$.

Tangent: $y - 1 = -4(x - 0) \Rightarrow \; y = -4x + 1$,

Normal $y - 1 = \frac{1}{4}(x - 0) \Rightarrow \; y = \frac{1}{4}x + 1$

(iii) At $x = 1$, $y = 0$, $m_T = 0$, $m_N =$ not defined.

Tangent: $y = 0$, Normal $x = 1$

11. (a) gradient is 0.6

(b) $y - \ln 5 = 0.6\,(x - 2)$ **OR** directly by GDC: $y = 0.6\,x + 0.409$

(c) at $x = 2$, $y = \ln 5$ (= 1.609...)
gradient of normal = – 5/3 = –1.6666...

normal: $y - \ln 5 = -\frac{5}{3}(x - 2)$ **OR** directly by GDC $y = -1.66666x + 4.94277$

For $y = 0$: $x = 2.97$ (accept 2.96)

coordinates of R are (2.97, 0)

12. Use GDC, Graph, Sketch tangent and Sketch normal

(a) $y = 1.92x - 1.92$

(b) $y = -0.52x + 0.520$

13.

(a) $f'(x) = \frac{1}{3x+1} \times 3 \quad \left(= \frac{3}{3x+1}\right)$

(b) Hence when $x = 2$, gradient of tangent $= \frac{3}{7}$

$\Rightarrow$ gradient of normal is $-\frac{7}{3}$

$y - \ln 7 = -\frac{7}{3}(x-2)$

$y = -\frac{7}{3}x + \frac{14}{3} + \ln 7$

(accept $y = -2.33x + 6.61$)

14. (a) At $x = a$, $h(x) = a^{\frac{1}{5}}$

$h'(x) = \frac{1}{5}x^{-\frac{4}{5}}$ => $h'(a) = \frac{1}{5a^{\frac{4}{5}}}$ = gradient of tangent

=> $y - a^{\frac{1}{5}} = \frac{1}{5a^{\frac{4}{5}}}(x-a) = \frac{1}{5a^{\frac{4}{5}}}x - \frac{1}{5}a^{\frac{1}{5}}$ => $y = \frac{1}{5a^{\frac{4}{5}}}x + \frac{4}{5}a^{\frac{1}{5}}$

(b) tangent intersects x-axis => $y = 0$ => $\frac{1}{5a^{\frac{4}{5}}}x = -\frac{4}{5}a^{\frac{1}{5}}$

=> $x = 5a^{\frac{4}{5}}\left(-\frac{4}{5}a^{\frac{1}{5}}\right) = -4a$

15. $y = x^2 - x \quad \frac{dy}{dx} = 2x - 1.$

Line parallel to $y = 5x \Rightarrow 2x - 1 = 5 \Rightarrow x = 3$ so $y = 6$

Point (3, 6)

16. gradient of tangent = 8

$f'(x) = 4kx^3$

$4kx^3 = 8 \Rightarrow kx^3 = 2$

substituting $x = 1$, $k = 2$

17. (a) $f'(x) = 6x - 5$

(b) $f'(p) = 7 \Rightarrow 6p - 5 = 7 \Rightarrow p = 2$

(c) Setting $y(2) = f(2)$

Substituting $y(2) = 7\times2 - 9 = 5$, **and** $f(2) = 3\times2^2 - 5\times2 + k = k + 2$

$k + 2 = 5 \Rightarrow k = 3$

18. (a) $f(1) = 3 \Rightarrow p + q = 3$

$f'(x) = 2px + q$

$f'(1) = 8 \Rightarrow 2p + q = 8$

$p = 5, q = -2$

(b) $f'(x) = 10x - 2$

$f'(0.2) = 0$, at $x = 0.2$ $y = -0.2$

Tangent $y = -0.2$ (horizontal line)

Normal $x = 0.2$ (vertical line)

19. (a) $f'(x) = -6 \sin 2x + 2 \sin x \cos x = -6 \sin 2x + \sin 2x = -5 \sin 2x$

(b) For $x = 0$, $y = 3$. gradient $m_T = 0$

Tangent line: $y = 3$

(c) $k = \frac{\pi}{2}$ $(=1.57)$

20. **METHOD 1**

(a) The equation of the tangent is $y = -4x - 8$.

(b) The point where the tangent meets the curve again is $(-2, 0)$.

METHOD 2

(a) $y = -4$ and $\frac{dy}{dx} = 3x^2 + 8x + 1 = -4$ at $x = -1$.

Therefore, the tangent equation is $y = -4x - 8$.

(b) **without GDC for HL only**

This tangent meets the curve when $-4x - 8 = x^3 + 4x^2 + x - 6$ which gives

$x^3 + 4x^2 + 5x + 2 = 0 \Rightarrow (x + 1)^2(x + 2) = 0$.

The required point of intersection is $(-2, 0)$.

21. For the curve, $y = 7$ when $x = 1 \Rightarrow a + b = 14$, and

$\frac{dy}{dx} = 6x^2 + 2ax + b = 16$ when $x = 1 \Rightarrow 2a + b = 10$.

Solving gives $a = -4$ and $b = 18$.

22. $\frac{dy}{dx} = -\frac{k}{x^2} + \frac{2}{x}$ When $x = 2$, gradient of normal $= -\frac{3}{2}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{3} \Rightarrow -\frac{k}{4} + 1 = \frac{2}{3} \Rightarrow k = \frac{4}{3}$

23.

METHOD 1

Line and graph intersect when $3x^2 - x + 4 = mx + 1$

i.e. $3x^2 - (1 + m)x + 3 = 0$

$y = mx + 1$ tangent to graph $\Rightarrow$

$3x^2 - (1 + m)x + 3 = 0$ has equal roots

i.e. $b^2 - 4ac = 0$

$\Rightarrow (1 + m)^2 - 36 = 0$

$\Rightarrow m = 5$, $m = -7$

METHOD 2

$f'(x) = 6x - 1$

$3x^2 - x + 4 = mx + 1$

Substitute $m = 6x - 1$ into above

$3x^2 - 6x^2 + 3 = 0$

$\Rightarrow x = \pm 1$

$\Rightarrow m = 5$, $m = -7$

24. **Method 1:** $y = 4 - x^2$ $\frac{dy}{dx} = -2x$

At the point of contact $2x = m$ and $mx + 5 = 4 - x^2$

Solving the system gives $m = \pm 2$.

Method 2: For intersection: $mx + 5 = 4 - x^2$ or $x^2 + mx + 1 = 0$.

For tangency: discriminant = 0

Thus, $m^2 - 4 = 0$, *so* $m = \pm 2$

25. $f(x)=5x^2+10$.

METHOD A:

A line passing through Q(1,10) has the form $y-10=m(x-1)\Rightarrow y=mx-m+10$

At the point of contact: Equal functions: $5x^2+10=mx-m+10$ (1)

Equal derivatives: $10x=m$ (2)

We solve the system:

(1) and (2): $5x^2+10=10x^2-10x+10 \Leftrightarrow 5x^2-10x=0 \Leftrightarrow 5x(x-2)=0 \Leftrightarrow x=0$ or $x=2$

From (2): If $x=0$, $m=0$, if $x=2$, $m=20$

Therefore If $m=0$ the tangent line is $y=10$

If $m=20$ the tangent line is $y=20x-10$

METHOD B:

$f(x)=5x^2+10$, $f'(x)=10x$.

At $x=a$, $y=5a^2+10$ (that is at point $(a,5a^2+10)$

$m_T=10a$

The general tangent line is $y-(5a^2+10)=10a(x-a)\Rightarrow y-5a^2-10=10ax-10a^2$

$\Rightarrow y=10ax-5a^2+10$

The line passes through Q(1,10), so

$10a-5a^2+10=10 \Leftrightarrow 10a-5a^2=0 \Leftrightarrow 5a(2-a)=0 \Leftrightarrow a=0$ or $a=2$

If $a=0$ the tangent line is $y=10$

If $a=2$ the tangent line is $y=20x-10$

26. $f(x)=4-x^2$.

METHOD A:

A line passing through (0,5) has the form $y-5=m(x-0)\Rightarrow y=mx+5$

At the point of contact: Equal functions: $4-x^2=mx+5$ (1)

Equal derivatives: $-2x=m$ (2)

We solve the system:

(1) and (2): $4-x^2=-2x^2+5 \Leftrightarrow x^2=1 \Leftrightarrow x=\pm1$

From (2): If $x=1$, $m=-2$, if $x=-1$, $m=2$

Therefore If $m=-2$ the tangent line is $y=2x+5$

If $m=2$ the tangent line is $y=-2x+5$

METHOD B:

$f(x)=4-x^2$, $f'(x)=-2x$.

At $x=a$, $y=4-a^2$ (that is at point $(a,4-a^2)$)

$m_T=-2a$

The general tangent line is $y-(4-a^2)=-2a(x-a)\Rightarrow y-4+a^2=-2ax+2a^2$

$\Rightarrow y=-2ax+a^2+4$

The line passes through (0,4), so

$a^2+4=5 \Leftrightarrow a^2=1 \Leftrightarrow a=\pm1$

If $a=1$ the tangent line is $y=-2x+5$

If $a=-1$ the tangent line is $y=2x+5$

B. Exam style questions (LONG)

27. (a) (i) $p = 1, q = 5$ (or $p = 5, q = 1$)

(ii) $x = 3$ (must be an equation)

(b) $y = (x - 1)(x - 5) = x^2 - 6x + 5 = (x - 3)^2 - 4$ ($h = 3, k = -4$)

(c) $\frac{dy}{dx} = 2(x-3)$ $(= 2x - 6)$

(d) When $x = 0$, $\frac{dy}{dx} = -6$

$y - 5 = -6(x - 0)$ ($y = -6x + 5$ or equivalent)

28. (a) $h = 3$ $k = 2$

(b) $f(x) = -(x-3)^2 + 2 = -x^2 + 6x - 9 + 2 = -x^2 + 6x - 7$

(c) $f'(x) = -2x + 6$

(d) (i) $m_T = -2$, $m_N = \frac{1}{2}$, Normal: $y - 1 = \frac{1}{2}(x-4) \Rightarrow y = \frac{1}{2}x - 1$

(ii) $-x^2 + 6x - 7 = \frac{1}{2}x - 1 \Leftrightarrow 2x^2 - 11x + 12 = 0 \Leftrightarrow$ $x = 1.5$ or $x=4$ so $x = 1.5$

(OR by GDC $-x^2 + 6x - 7 = \frac{1}{2}x - 1 \Rightarrow x = 1.5$

29. (a) (i) $f'(x) = -x + 2$

(ii) $f'(0) = 2$

(b) Gradient of tangent at y-intercept $m_T = f'(0) = 2$, $m_N = -\frac{1}{2}$

Therefore, equation of the normal is $y - 2.5 = -0.5(x - 0) \Rightarrow y = -0.5x + 2.5$

(c) (i) $-0.5x^2 + 2x + 2.5 = -0.5x + 2.5 \Rightarrow x = 0$ or $x = 5$

(ii) Curve and normal intersect when $x = 0$ or $x = 5$
Other point is when $x = 5 \Rightarrow y = -0.5(5) + 2.5 = 0$ (so other point (5, 0))

30. (a) (i) $p = -2$ $q = 4$ (or $p = 4, q = -2$)

(ii) $y = a(x+2)(x-4) \Leftrightarrow 8 = a(6+2)(6-4) \Leftrightarrow 8 = 16a \Leftrightarrow a = \frac{1}{2}$

(iii) $y = \frac{1}{2}(x+2)(x-4) \Rightarrow y = \frac{1}{2}(x^2 - 2x - 8) \Rightarrow y = \frac{1}{2}x^2 - x - 4$

(b) $\frac{dy}{dx} = x - 1$

$x - 1 = 7 \Leftrightarrow x = 8, y = 20$ (P is (8, 20))

(c) (i) when $x = 4$, $m_T = 4 - 1 = 3 \Rightarrow m_N = -\frac{1}{3}$

$y - 0 = -\frac{1}{3}(x-4)$ $\left(y = -\frac{1}{3}x + \frac{4}{3}\right)$

(ii) $\frac{1}{2}x^2 - x - 4 = -\frac{1}{3}x + \frac{4}{3} \Leftrightarrow x = -\frac{8}{3}$ or $x = 4$

$x = -\frac{8}{3}$ (−2.67)

31. (a) (i) $f(x) = \dfrac{2x+1}{x-3} = 2 + \dfrac{7}{x-3}$ OR $f(x) = \dfrac{2+\frac{1}{x}}{1-\frac{3}{x}}$

Therefore as $|x| \to \infty$ $f(x) \to 2$ $\Rightarrow y = 2$ is an asymptote

Note: *inexact methods based on the ratio of the coefficients of x also accepted*

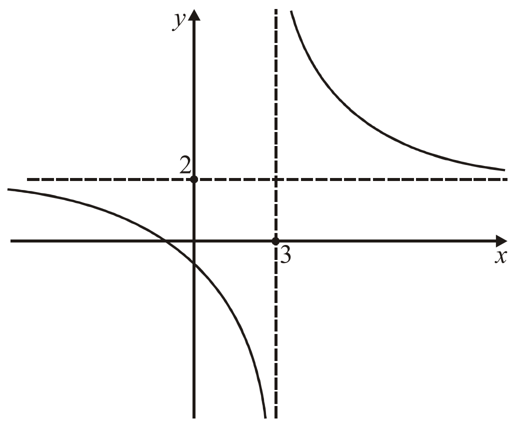
(ii) Asymptote at $x = 3$

(iii) $P(3, 2)$

(b) $f(x) = 0 \Rightarrow x = -\dfrac{1}{2}$ $\left(-\dfrac{1}{2}, 0\right)$

$x = 0 \Rightarrow f(x) = -\dfrac{1}{3}$ $\left(0, -\dfrac{1}{3}\right)$

(c)



(d) $f'(x) = \dfrac{(x-3)(2)-(2x+1)}{(x-3)^2} = \dfrac{-7}{(x-3)^2}$

Therefore slope when $x = 4$ is -7

And $f(4) = 9$ ie $S(4, 9)$

Equation of tangent: $y - 9 = -7(x - 4) \Rightarrow 7x + y - 37 = 0$

(e) at T, $\dfrac{-7}{(x-3)^2} = -7 \Rightarrow (x-3)^2 = 1 \Rightarrow x - 3 = \pm 1$

$\left.\begin{array}{l} x = 4 \text{ or } 2 \\ y = 9 \text{ or } -5 \end{array}\right\}$ $\begin{array}{l} S(4, 9) \\ T(2, -5) \end{array}$

(f) Midpoint $[ST] = \left(\dfrac{4+2}{2}, \dfrac{9-5}{2}\right) = (3, 2) =$ point P

32. (a) $f'(x) = 3x^2 - 6x - 24$

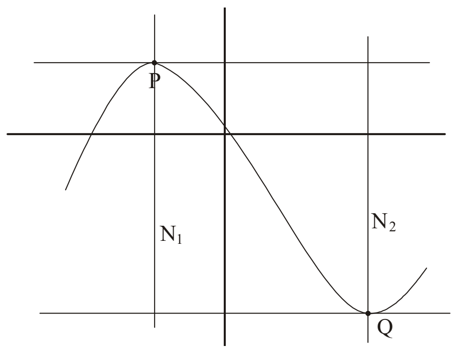
(b) Tangents parallel to the x-axis mean maximum and minimum (see graph)

EITHER by GDC P(–2, 29) and Q(4, –79)

OR $f'(x) = 0 \Leftrightarrow 3x^2 - 6x - 24 = 0 \Leftrightarrow x = -2$ or $x = 4$

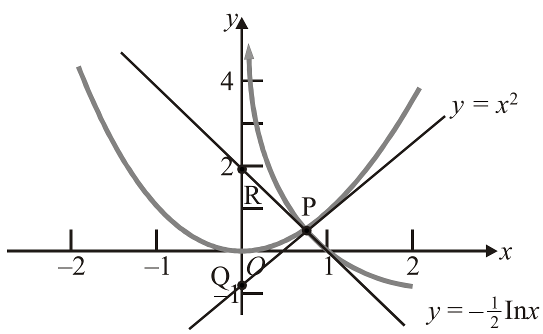
Coordinates are P(–2, 29) and Q(4, –79)

(b)



(i) (4, 29) (ii) (–2, –79)

33. (a)



(b) $x^2 + \frac{1}{2}\ln x = 0$ when $x = 0.548217$. Therefore, the x-coordinate of P is 0.548….

(c) The tangent at P to $y = x^2$ has equation $y = 1.0964x - 0.30054$,

and the tangent at P to $y = -\frac{1}{2}\ln x$ has equation $y = -0.91205x + 0.80054$.

Thus, the area of triangle PQR = $\frac{1}{2}(0.30052 + 0.80054)(0.5482) = 0.302$ (3 s.f.)

(d) $y = x^2 \Rightarrow$ when $x = a$, $\frac{dy}{dx} = 2a$

$y = -\frac{1}{2}\ln x \Rightarrow$ when $x = a$, $\frac{dy}{dx} = -\frac{1}{2a}$ $(a > 0)$

Now, $(2a)\left(-\frac{1}{2a}\right) = -1$ for all $a > 0$.

Therefore, the tangents to the curve at $x = a$ on each curve are always perpendicular.

34. (a) **EITHER** $A\sin\left(\frac{\pi}{2}\right) + B = 3$ and $A\sin\left(\frac{3\pi}{2}\right) + B = -1$

$\Leftrightarrow A + B = 3, -A + B = -1$

$\Leftrightarrow A = 2, B = 1$

OR

Amplitude $= A = \frac{3-(-1)}{2} = \frac{4}{2} = 2$

Midpoint value $= B = \frac{3+(-1)}{2} = \frac{2}{2} = 1$

(b) $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$

$f'(x) = \left(\frac{\pi}{2}\right)2\cos\left(\frac{\pi}{2}x\right) + 0 = \pi\cos\left(\frac{\pi}{2}x\right)$

(c) (i) $y = k - \pi x$ is a tangent $\Rightarrow -\pi = \pi\cos\left(\frac{\pi}{2}x\right)$

$\Rightarrow -1 = \cos\left(\frac{\pi}{2}x\right)$

$\Rightarrow \frac{\pi}{2}x = \pi$ or 3π or ...

$\Rightarrow x = 2$ or 6 ...

Since $0 \le x \le 5$, we take $x = 2$, so the point is $(2, 1)$

(ii) Tangent line is: $y = -\pi(x-2) + 1$

$y = (2\pi + 1) - \pi x$

$k = 2\pi + 1$

(d) $f(x) = 2 \Rightarrow 2\sin\left(\frac{\pi}{2}x\right) + 1 = 2$

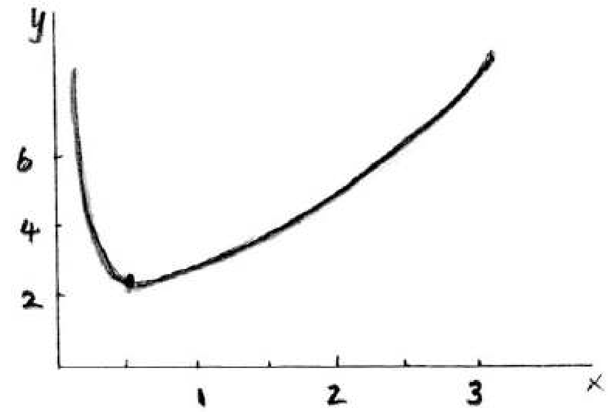
$\Rightarrow \sin\left(\frac{\pi}{2}x\right) = \frac{1}{2}$

$\Rightarrow \frac{\pi}{2}x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{13\pi}{6}$

$x = \frac{1}{3}$ or $\frac{5}{3}$ or $\frac{13}{3}$

35.

(a) (i)



(ii) $g(x)=\dfrac{e^x}{\sqrt{x}}$

$$g'(x)=\frac{e^x\sqrt{x}-\dfrac{e^x}{2\sqrt{x}}}{x}$$

$$=\frac{(2x-1)e^x}{2x\sqrt{x}}$$

(iii) gradient is $-\dfrac{1}{g'(x)}$

$$=\frac{2x\sqrt{x}}{(1-2x)e^x}$$

(b) (i) $\dfrac{y-0}{x-1}=\dfrac{e^x}{\sqrt{x}(x-1)}$

(ii) **EITHER**

$$\frac{e^x}{\sqrt{x}(x-1)}=\frac{2x\sqrt{x}}{(1-2x)e^x}$$

$x=0.5454\ ...$

OR

$$D^2=(x-1)^2+y^2=(x-1)^2+\frac{e^{2x}}{x}$$

$$\frac{dD^2}{dx^2}=2(x-1)+\frac{2e^{2x}x-e^{2x}}{x^2}=0$$

$x=0.5454\ ...$

THEN

$$\text{distance}=\sqrt{(1-0.5454)^2+\left(\frac{e^{0.5454}}{\sqrt{0.5454}}\right)^2}$$

$=2.38$