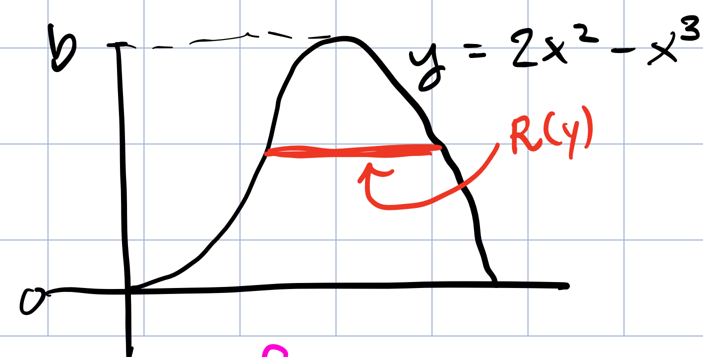
Class 5

§6.3 Cylindrical Shells

- TEST 1 in 1 Week
  (on Valentine's Day ♡)

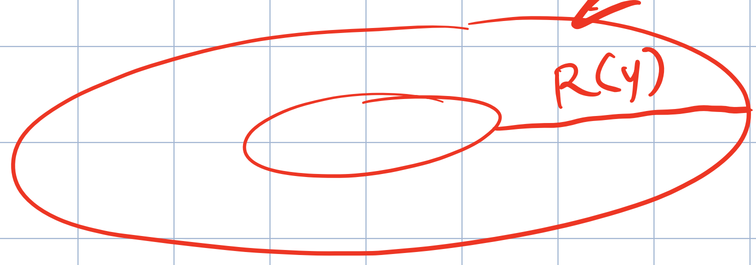
- Review Session at Math Center
  2/13/24, 3pm-4:30pm

Ex: Compute the volume of the solid obtained by rotating $y = 2x^2 - x^3$, for $0 \le x \le 2$, about the y-axis:



Try to compute the area of the washer at height y:

area $A(y)$



- this is a hassle

$$\int_0^b A(y)\,dy$$

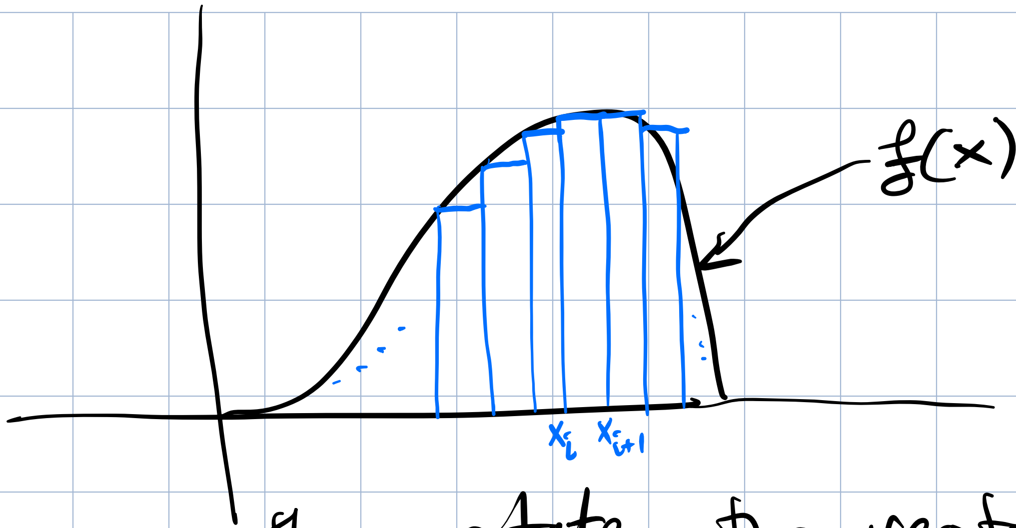
finding b requires differentiating $y=2x^2-x^3$

finding area requires solving $y=2x^2-x^3$ for x

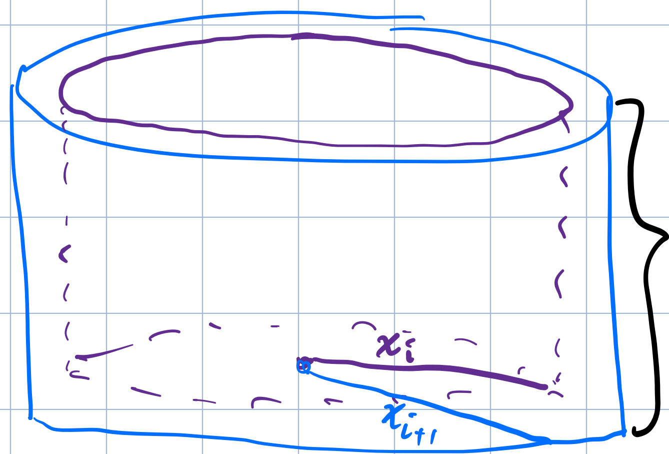
Alternative Approach:

Approximate the solid with "cylindrical shells"

- subdivide the region under the function into rectangles (approximation)



-then rotate the rectangles



$h \approx f(x_i)$

$\approx f\left(\frac{x_{i+1}+x_i}{2}\right)$

Volume of this "cylindrical shell" is

$$V = \pi x_{i+1}^2 h - \pi x_i^2 h$$

$$= \pi h \left(x_{i+1}^2 - x_i^2\right)$$

$$= \pi h \left(x_{i+1} + x_i\right)\overbrace{\left(x_{i+1} - x_i\right)}^{\Delta x}$$

$$= 2\pi h \Delta x \, \frac{x_{i+1} + x_i}{2}$$

$$\approx 2\pi \underbrace{f\left(\frac{x_{i+1}+x_i}{2}\right)} \cdot \Delta x \cdot \underbrace{\frac{x_{i+1}+x_i}{2}}$$

$\hookrightarrow f(x)$ $\hookrightarrow x$

let $\Delta x \to 0$ (or let the number of partitions go to ∞)

This gives the volume of the solid as:

$$\int_0^2 2\pi \cdot x \cdot f(x)\, dx$$

$$= \int_0^2 2\pi x (2x^2 - x^3)\, dx = \cdots = \frac{16\pi}{5}$$

General Formula

The volume of the solid obtained by rotating $f(x)$, with $0 \le a \le x \le b$, about the y-axis, is

$$\int_a^b 2\pi x f(x)\, dx$$

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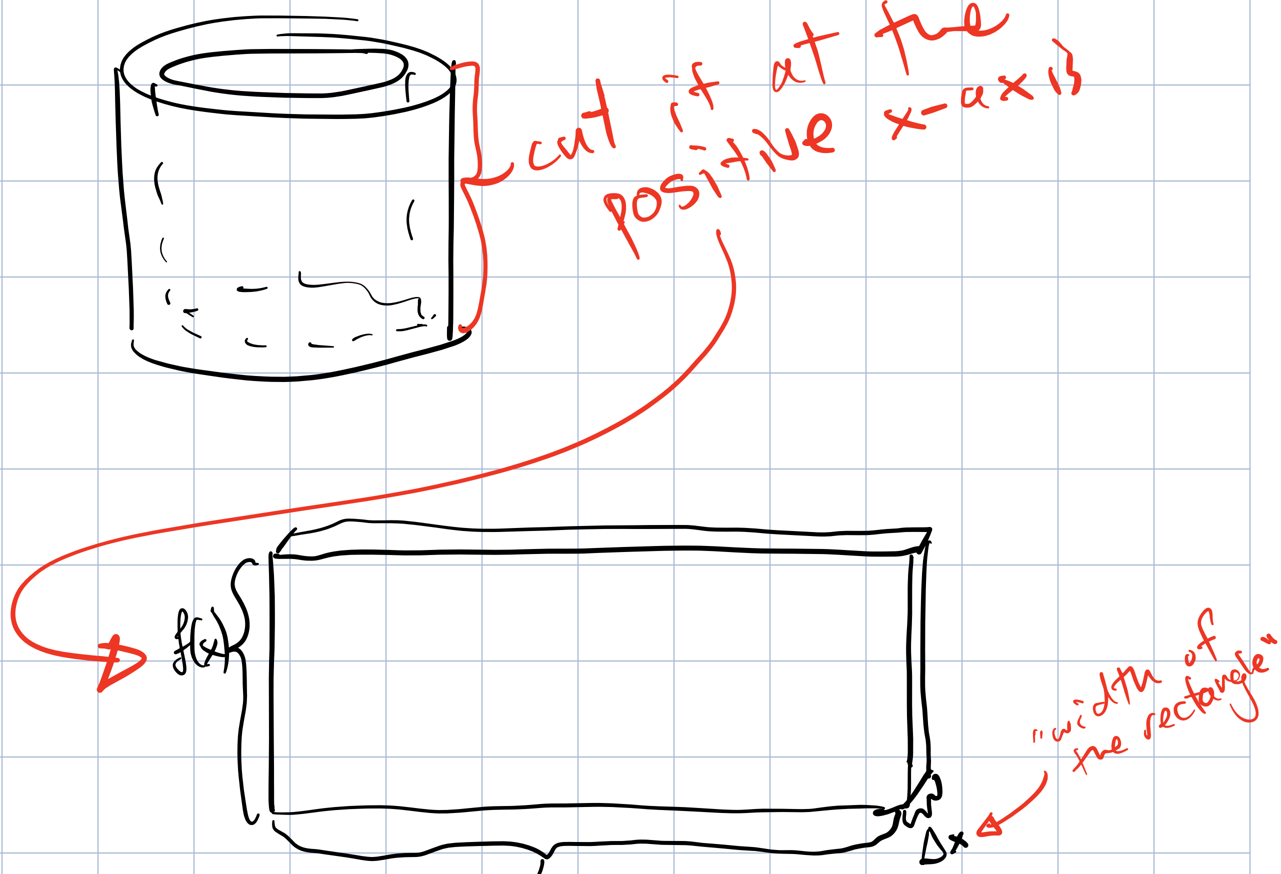
ab:

Rema

To help remember the formula
"unwrap the cylinder"



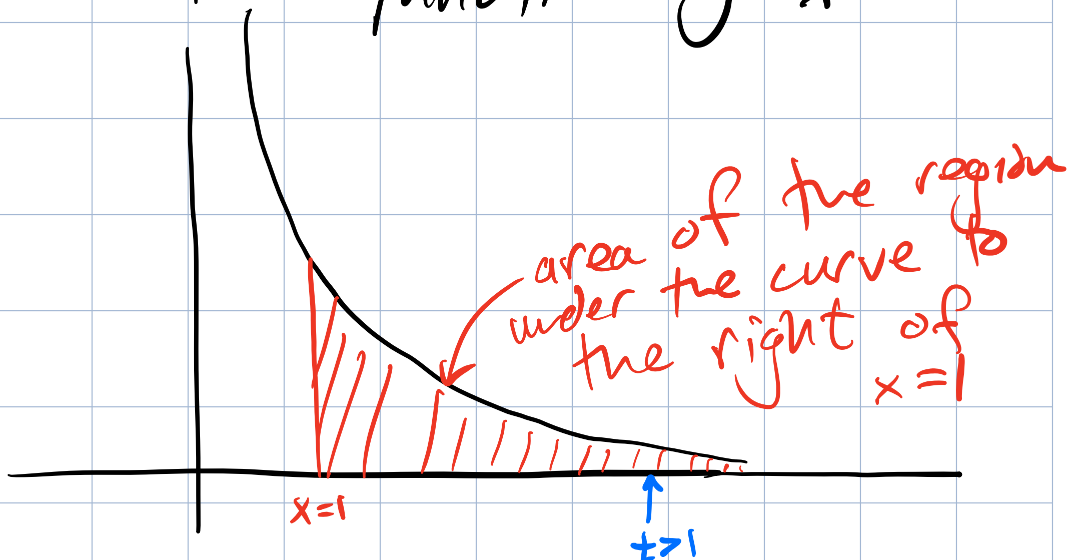
circumference of the inner circle $2\pi x$

volume of this is $2\pi x \cdot f(x) \cdot \overset{dx}{\cancel{\Delta x}}$

For you to read: p. 463
"Disks and Washers vs. Shells"

§7.8: Improper Integrals

Ex: Consider the function $y=\frac{1}{x^2}$

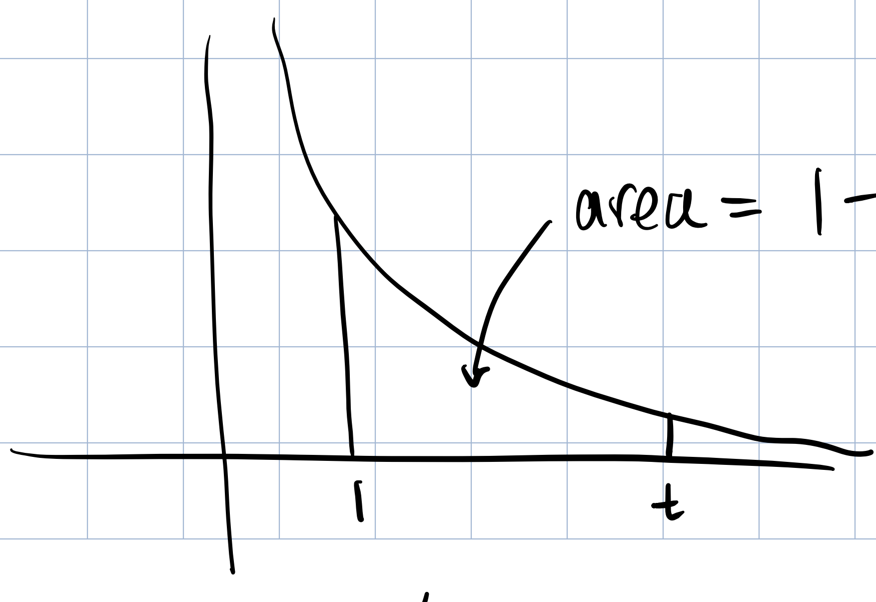


Seems like it should be infinite!

Consider

$$\int_1^t \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^t = \frac{-1}{t}+\frac{1}{1} = 1-\frac{1}{t}$$

So

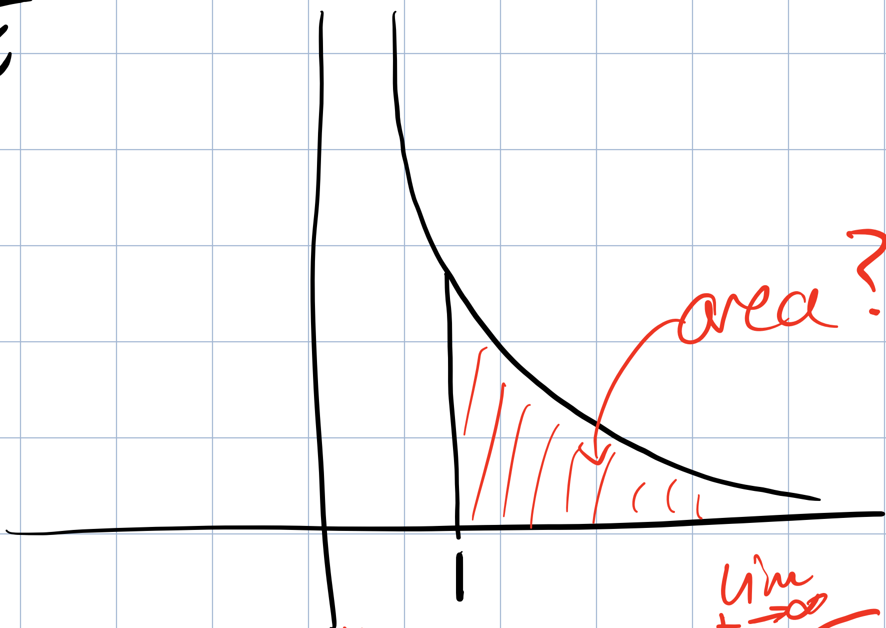


Consider $\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left(1-\frac{1}{t}\right)$

$$= 1 - 0 = 1$$

counter intuitive!

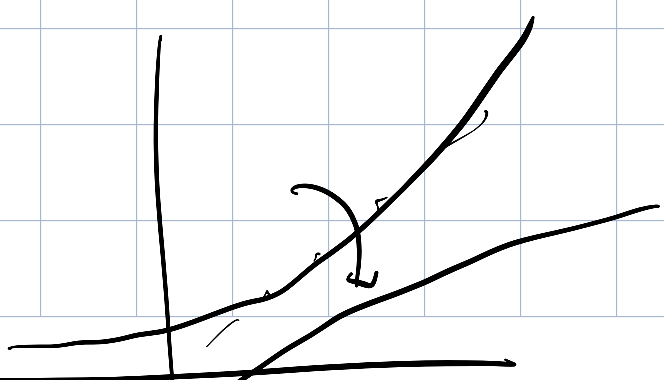
Notice: Do the same thing with $\frac{1}{x}$

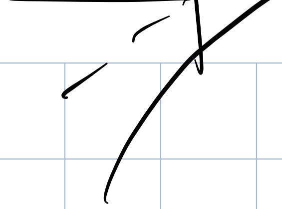


$$\lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \underbrace{\left[\ln(x)\right]_1^t}_{\lim_{t\to\infty}} = \underbrace{\ln(t)-\ln(1)}_{\lim_{t\to\infty}}$$

$$= \underbrace{\ln(t)}_{\lim_{t\to\infty}}$$

diverges!





$\ln(t)$ is supposed to be the inverse of e^t $\Rightarrow$ inverse graphs can be sketched by reflecting across the $y=x$ line

Definition of an Improper Integral of Type I

(i) $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

(ii) $$\int_{-\infty}^a f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

(iii) $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

for any a in $\mathbb{R}$

For (i) and (ii) if that limit exists we say the integral converges. Otherwise we say it diverges. We say (iii) converges only when <u>both</u> of the improper integrals on the RHS converge.

<u>Improper Integrals of Type II</u>

(i) Say $f(x)$ is cont. on $(a, b]$ and discontinuous at a. Then we define

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

(ii) Say $f(x)$ is cont. on $[a, b)$ but disc. at b. Then we define

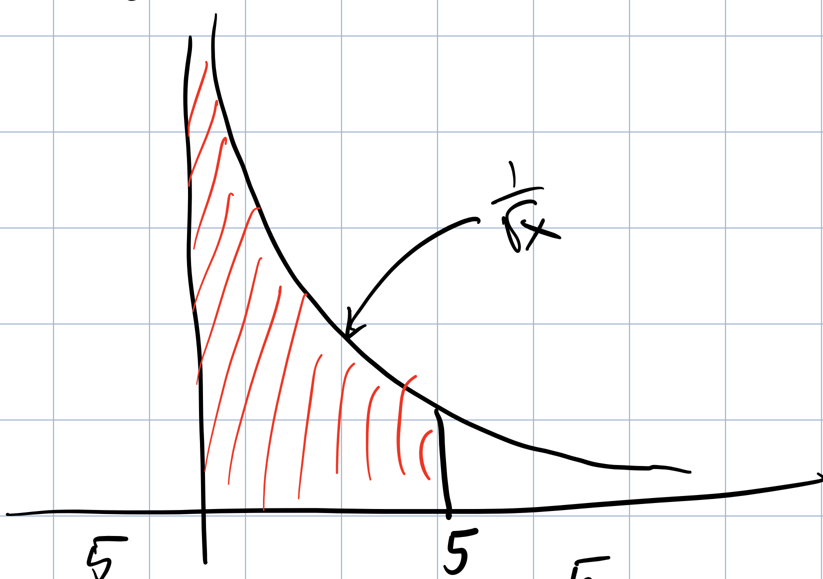
$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

(iii) Say f is disc. at c but cont. on $[a, c)$ and $(c, b]$
Then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Same terminology for convergence/divergence

Ex: Integrate $\frac{1}{\sqrt{x}}$ from 0 to 5

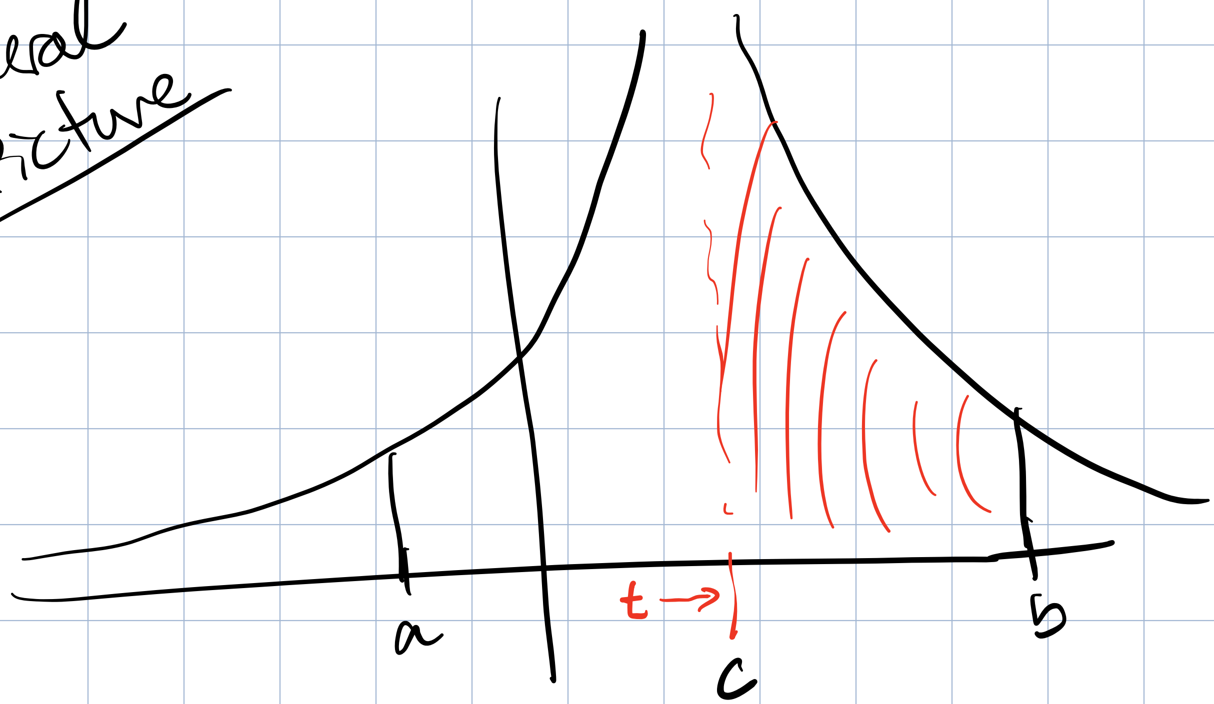


This is $\displaystyle\int_0^5 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^5 x^{-\frac{1}{2}}\,dx$

$$= \lim_{t \to 0^+} \left[2x^{\frac{1}{2}} \right]_t^5 = 2\sqrt{5} - \overbrace{\lim_{t \to 0^+} 2\sqrt{t}}^{0}$$

$$= 2\sqrt{5}$$

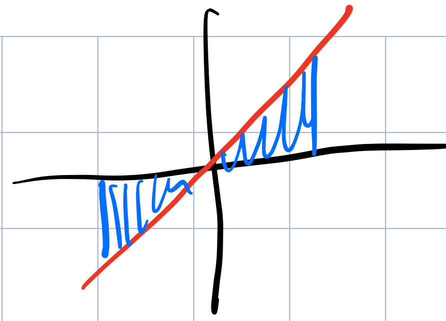
General Picture



Why for Type I part (iii) we can't do

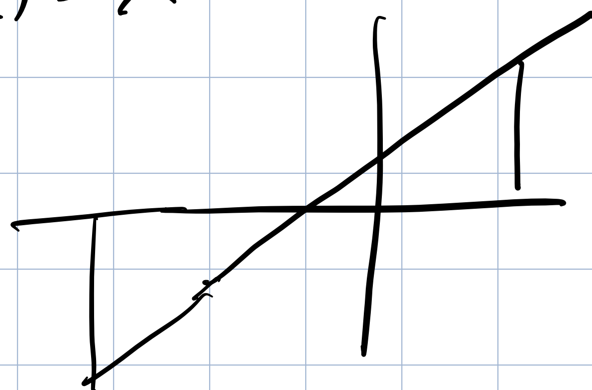
$$\lim_{t \to \infty} \int_{-t}^{t} f(x)\,dx \ ?$$

Ex: Say $f(x)=x$



$$\lim_{t\to\infty}\int_{-t}^{t} x\,dx = 0$$

Ex: Say $f(x)=x+1$



But

$$\lim_{t\to\infty}\int_{-t}^{t} x+1\,dx = \frac{1}{2}x^2+x\Big|_{-t}^{t}$$

$$= \overbrace{\tfrac{1}{2}t^2+t}^{\lim} - \overbrace{\tfrac{1}{2}t^2-(-t)}^{\lim} = 2t$$

This diverges!