Exponents and Radicals in Algebra

Simplifying expression with coefficients and exponents:

Basic concepts of operating with term having **coefficients and exponents**:

A single variable has a coefficient of 1 and an exponent of 1: $x = 1 \cdot x^1$.
Note: When you have an expression in parenthesis: $(3x + 5y) = \mathbf{1} \cdot (3x + 5y)^{\mathbf{1}}$
And if there is a negative sign, it means: $-x = -1 \cdot x^1$; $-(3x + 5y) = \mathbf{-1} \cdot (3x + 5y)^{\mathbf{1}}$

Addition (coefficients add, exponents must be the same; variable parts same)
$3x^4 + 3x^4 = 6x^4$ $\quad$ $7x^4 + 3x^4 = 10x^4$

Subtraction (coefficients subtract, exponents must be the same; variable parts same)
$3x^4 - 3x^4 = 0$ $\quad$ $7x^4 - 3x^4 = 4x^4$

Multiplication (coefficients multiply, exponents add like variables)
$3x^4 \cdot 3x^4 = 9x^8$ $\quad$ $7x^5 \cdot 3x^4 = 21x^9$ $\quad$ $5x^2y^3 \cdot 4x^3y^5 = 20x^5y^8$

Division (coefficients divide, exponents subtract like variable)
$3x^4 \div 3x^4 = 1$ $\quad$ $21x^9 \div 3x^4 = 7x^5$ $\quad$ $20x^5y^8 \div 4x^3y^5 = 5x^2y^3$

Solving Linear Equations:

When solving equations, you may have multiple choices as your first step; it is suggested to always simplify the variable parts first. So that the variable's coefficient is positive.

This section covers:
- **Introducing Exponents and Radicals (Roots) with Variables**
- **Properties of Exponents and Radicals**, **Putting Exponents and Radicals in the Calculator**
- **Rationalizing Radicals**
- **Simplifying Exponential Expressions**
- **Solving Exponential and Radical Equations**
- **Solving Simple Radical Inequalities**
- **More Practice**

We briefly talked about exponents in the **Powers, Exponents, Radicals (Roots) and Scientific Notation** section, but we need to go a little bit further in depth and talk about how to do algebra with them. Note that we'll see more radicals in the **Solving Radical Equations and Inequalities** section, and we'll talk about **Factoring with Exponents**, and **Exponential Functions** in the **Exponential Functions** section.

Remember that **exponents**, or "raising" a number to a power, are just the number of times that the number (called the **base**) is multiplied by itself.

Radicals (which comes from the word "root" and means the same thing) means undoing the exponents, or finding out what numbers **multiplied by themselves** comes up with the number.

We remember that $\sqrt{25} = 5$, since $5 \times 5 = 25$. Note that we have to remember that when taking the square root (or any even root), we always take the **positive value** (just memorize this).

Introducing Exponents and Radicals (Roots) with Variables

But now that we have learned some algebra, we can do exponential problems with variables in them! We have $x^2 = x$ (actually $x^2 = |x|$ since x can be negative) since $x \times x = x^2$. We also learned that taking the square root of a number is the same as raising it to $\frac{1}{2}$, so $x^{\frac{1}{2}} = \sqrt{x}$. Also, remember that when we take the square root, there is an invisible **2** in the radical, like this: $\sqrt[2]{x}$.

Also note that what is under the radical sign is called the **radicand** (x in the previous example), and for the n**th root**, the **index** is n (**2**, in the previous example, since it's a square root).

With a **negative exponent**, there is nothing to do with negative numbers! You move the base from the numerator to the denominator (or denominator to numerator) and make it positive! If you have a base with a negative number that is not a fraction, put **1** over it and make the exponent positive.

If the negative exponent is on the outside parentheses of a fraction, take the **reciprocal** of the fraction (base) and make the exponent positive. Some examples: $x^{-2} = \left(\frac{1}{x}\right)^2$ and $\left(\frac{y}{x}\right)^{-4} = \left(\frac{x}{y}\right)^4$.

Just a note that we're only dealing with **real numbers** at this point; later we'll learn about **imaginary numbers**, where we can take the square root of a negative number using a special symbol, $\boldsymbol{i}$.

Properties of Exponents and Radicals

Remember these **basic rules**:

Exponents and Radical Rules	Example	Notes
$x^m = x \cdot x \cdot x \cdot x \cdot \ldots (m\ times)$	$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	x is the base; m is the exponent
$\sqrt[m]{x} = y\ means\ y^m = x$	$\sqrt[3]{8} = 2\ since$ $2 \cdot 2 \cdot 2 = 2^3 = 8$	y is the base; m is the root
$\sqrt{x} = \sqrt[2]{x}$	$\sqrt{9} = 3\ means\ \sqrt[2]{9} = 3$	The default root is **2** (square root).
$x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$	$x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$	If a root is raised to a fraction (rational), the numerator of the exponent is the power and the denominator is the root. When raising a radical to an exponent, the exponent can be on the "inside" or "outside".
$x^{-m} = \dfrac{1}{x^m} \qquad \dfrac{1}{x^{-m}} = x^m$ $\left(\dfrac{x}{y}\right)^{-m} = \left(\dfrac{y}{x}\right)^m$	$2^{-3} = \dfrac{1}{2^3};\ \dfrac{1}{2^{-3}} = 2^3 = 8$ $\left(\dfrac{3}{4}\right)^{-3} = \left(\dfrac{4}{3}\right)^3 = \dfrac{64}{27}$	Raising a base to a **negative exponent** means taking the **reciprocal** and making the exponent positive.
$a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$ (Doesn't work for imaginary numbers under radicals)	$3\sqrt{5} \times 6\sqrt{7} = 18\sqrt{35}$	When you multiply two radical terms, you can multiply what is on the outside, and also what's in the inside. You can only do this if the **roots are the same** (like square root, cube root).
$\sqrt[even]{negative\ number}$ exists for **imaginary numbers**, but not for **real numbers**.	$\sqrt[4]{-81}$ = **no real solution**	We cannot take the even root of a negative number and get a real number. We can get an "imaginary number", which we will see later.
$\sqrt[odd]{x^{odd}} = x$ $\sqrt[even]{x^{even}} = \lvert x \rvert$	$\sqrt[3]{(-3)^3} = \sqrt[3]{-27} = -3$ $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$	A root "undoes" raising a number to that exponent. **For an even root, we only take positive value, even if original was negative.** For example, we squared **–2** under the square root, but our answer is **2**, which is $\lvert -2 \rvert$ (the absolute value of **2**).
For $y = x^{even}, y = \pm\sqrt[even]{x}$	$x^4 = 16;\ x = \pm 2$	Since we are taking an **even root**, we have to include both the **positive and negative solutions** in an **equation** with an even exponent. Remember that the square root sign only gives you the positive solutions. This is because **both** the positive root and negative roots work, when raised to that even power.

In algebra, we'll need to know these and many other basic rules on how to handle exponents and roots when we work with them. Here are the rules/properties with explanations and examples. In the "proof" column, you'll notice that we're using many of the algebraic properties that we learned in the **Types of Numbers and Algebraic Properties** section, such as the Associate and Commutative properties.

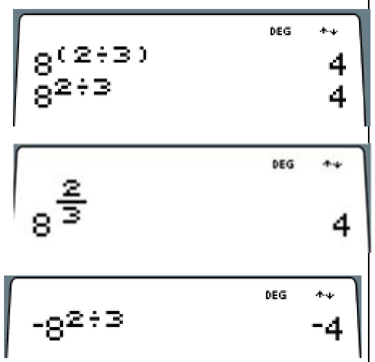
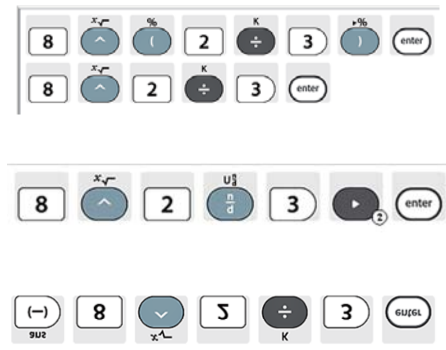
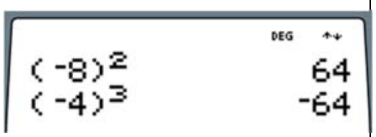
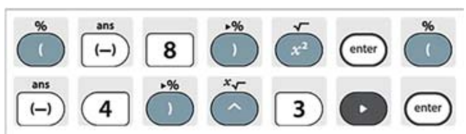
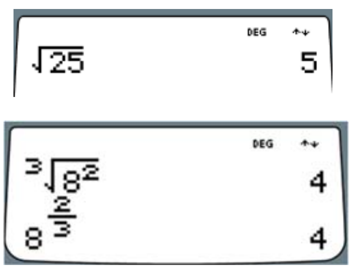
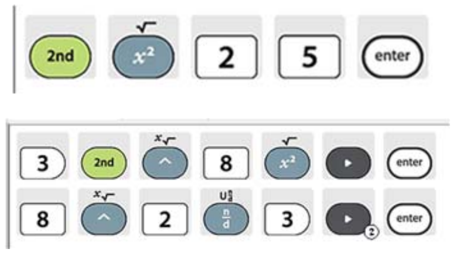
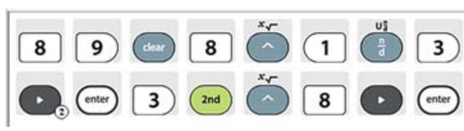
Unless otherwise indicated, **assume numbers under radicals with even roots are positive, and numbers in denominators are nonzero**.

https://www.shelovesmath.com/algebra/intermediate-algebra/exponents-and-radicals-roots/

Exponential/Radical Property	Example	"Proof" or an Explanation
$(xy)^m = x^m \cdot y^m$	$(xy)^3 = x^3 \cdot y^3$	$(xy)^3 = xy \cdot xy \cdot xy = (x \cdot x \cdot x) \cdot (y \cdot y \cdot y) = x^3 \cdot y^3$
$\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$	$\left(\frac{x}{y}\right)^5 = \frac{x^5}{y^5}$	$\left(\frac{x}{y}\right)^5 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x \cdot x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y \cdot y} = \frac{x^5}{y^5}$
$x^m \cdot x^n = x^{m+n}$	$x^3 \cdot x^4 = x^7$	$x^3 \cdot x^4 = (x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x)$ $= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^{3+4} = x^7$
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{x^7}{x^5} = x^2$	$\frac{x^7}{x^5} = \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = x \cdot x = x^{7-5} = x^2$
$(x^m)^n = x^{mn}$	$(x^3)^4 = x^{12}$	$(x^3)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = x^{3 \cdot 4}$ $= (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x^{12}$
$x^1 = x$	$(124{,}576{,}398)^1$ $= 124{,}576{,}398$	x is not multiplied by anything, so it is just x.
$x^0 = 1, if\ x \neq 0$	$124{,}576{,}398^0 = 1$	$1 = \frac{x^8}{x^8} = x^{8-8} = x^0$
$\frac{1}{x^m} = x^{-m}$ $\left(\frac{x}{y}\right)^{-m} = \left(\frac{y}{x}\right)^m$	$\frac{1}{3^2} = 3^{-2} = \frac{1}{9}$ $\left(\frac{3}{5}\right)^{-3} = \left(\frac{5}{3}\right)^3 = \frac{125}{27}$	$\frac{1}{3^2} = \frac{3^0}{3^2} = 3^{0-2} = 3^{-2}$ $\left(\frac{3}{5}\right)^{-3} = \frac{3^{-3}}{5^{-3}} = \frac{\frac{1}{3^3}}{\frac{1}{5^3}} = \frac{\frac{1}{27}}{\frac{1}{125}} = \frac{1}{27} \times \frac{125}{1} = \frac{125}{27}$ $\left(\frac{x}{y}\right)^{-m} = \frac{x^{-m}}{y^{-m}} = \frac{\frac{1}{x^m}}{\frac{1}{y^m}} = \frac{1}{x^m} \times \frac{y^m}{1} = \left(\frac{y}{x}\right)^m$
$\sqrt[n]{x} = x^{\frac{1}{n}}$	$\sqrt[3]{27} = 27^{\frac{1}{3}}$	The nth root of a base can be written as that base raised to the reciprocal of n, or $\frac{1}{n}$.
$\sqrt[m]{xy} = \sqrt[m]{x} \cdot \sqrt[m]{y}$	$\sqrt[3]{532} = \sqrt[3]{2 \cdot 8 \cdot 27}$ $= \sqrt[3]{2} \cdot \sqrt[3]{8} \cdot \sqrt[3]{27}$ $= \sqrt[3]{2} \cdot 2 \cdot 3 = 6\sqrt[3]{2}$	$\sqrt[3]{xy} = (xy)^{\frac{1}{3}} = x^{\frac{1}{3}} \cdot y^{\frac{1}{3}} = \sqrt[3]{x} \cdot \sqrt[3]{y}$ (Does not work for imaginary numbers under radicals.)
$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$	$\sqrt[3]{\frac{27}{8}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}$	$\sqrt[3]{\frac{x^3}{y^3}} = \frac{\sqrt[3]{x \cdot x \cdot x}}{\sqrt[3]{y \cdot y \cdot y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{y}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{y}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{y}} = \frac{x}{y} = \frac{\sqrt[3]{x^3}}{\sqrt[3]{y^3}}$
$\left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m} = x^{\frac{m}{n}}$ (if n is even, $x \geq 0$)	$8^{\frac{2}{3}} = \sqrt[3]{8^2}$ $= \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$	You can take an expression in radical form and turn it into rational form by using a fractional exponent with the original exponent on top and root on bottom (remember: the root is in a "cave" so it needs to go on the bottom)
n is **odd**: $\left(\sqrt[n]{x}\right)^n = \sqrt[n]{x^n} = x$	$\left(\sqrt[3]{-2x}\right)^3 = \sqrt[3]{(-2)^3}$ $= \sqrt[3]{-8} = -2$	$\left(\sqrt[5]{x}\right)^5 = \sqrt[5]{x^5} = x^{\frac{5}{5}} = x^1 = x$ Note that this works when n is even too, if $x \geq 0$.
n is **even**: $\sqrt[n]{x^n} = \|x\|$	$\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2$ $\sqrt[4]{2^4} = \sqrt[4]{16} = 2$	For when x is a negative: $\sqrt[4]{(neg\ number\ x)^4} = \sqrt[4]{pos\ number\ x^4} = pos\ x = \|x\|$
$\frac{x}{\sqrt{y}} = \frac{x}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y}$	$\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2$	This is called "**rationalizing**" the denominator (getting rid of the radical in the denominator) and is considered better "grammar" in math.

I know this seems like a **lot** to know, but after a lot of practice, they become second nature. You will have to learn the basic properties, but after that, the rest of it will fall in place!

Using a TI30 XS Multiview Calculator

Screen	Keystrokes	Notes
		To put a radical in the calculator, we can type "8^(2/3)". Notice, some calculators need the parenthesis around $\frac{2}{3}$ as the calculator follows the order of operation and may square 8 and then divide by 3. On our calculator this is not required. Notice when we place a negative on the outside of the 8, it performs the radicals first and puts the negative in front of it. You need to know your calculator.
		When the number raised to any power is a negative value, the negative **must be** in a set of **parenthesis**. This true for all built-in functions of the TI-30 XS
		We can take a square root by using the 2nd button, and the x^2 button. We can take the n^{th} roots with the 2nd button, and the ^ button. The cube root of 8^2. Shown are two ways to do the same operation.
		You can take the n^{th} root of any number by raising the number to the $\frac{1}{n}$.

Rationalizing Radicals

Before we work example, let us talk about **rationalizing radical fractions**. In math, sometimes we have to worry about "proper grammar". When radicals, it is improper grammar to have a root on the bottom in a fraction – in the denominator. To fix this, we multiply by a fraction with the bottom radical(s) on both the top and bottom (so the fraction equals **1**); this way the bottom radical disappears. Neat trick!

Here are some examples:

Math	Notes
$\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{1\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}=\frac{\sqrt{2}}{2}$	Since the $\sqrt{2}$ is on the bottom, we need to get rid of it by multiplying by 1, or $\frac{\sqrt{2}}{\sqrt{2}}$.
$\frac{4}{2\sqrt{3}}=\frac{4}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{4\sqrt{3}}{2\sqrt{3}\cdot\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}=\frac{2\sqrt{3}}{3}$	Since the $\sqrt{3}$ is on the bottom, we need to multiplying by 1, or $\frac{\sqrt{3}}{\sqrt{3}}$. Note that we did not need to multiply by $2\sqrt{3}$, only by the radical.
$\frac{5}{2\sqrt[4]{3}}=\frac{5}{2\sqrt[4]{3}}\cdot\frac{\left(\sqrt[4]{3}\right)^3}{\left(\sqrt[4]{3}\right)^3}=\frac{5\left(\sqrt[4]{3}\right)^3}{2\sqrt[4]{3}\left(\sqrt[4]{3}\right)^3}=\frac{5\left(\sqrt[4]{3}\right)^3}{2\left(\sqrt[4]{3}\right)^4}$ $=\frac{5\left(\sqrt[4]{3}\right)^3}{2\cdot 3}=\frac{5\left(\sqrt[4]{3}\right)^3}{6}$	Since we have the 4th root of **3** on the bottom ($\sqrt[4]{3}$), we can multiply by **1**, with the numerator and denominator being that radical **cubed**, to eliminate the 4th root. Do not worry if you do not totally get this now!

$$\sqrt[n]{x}=x^{\frac{1}{n}}$$

$$\sqrt[n]{x^m}=x^{\frac{m}{n}}$$

$$\sqrt[5]{x^{10}}=x^{\frac{10}{5}}=x^2$$

Simplifying Exponential Expressions

There are five main things you'll have to do to simplify exponents and radicals. For the purpose of the examples below, we are assuming that **variables in radicals are non-negative, and denominators are nonzero**.

- **get rid of parentheses ()**. Remember that when an exponential expression is raised to another exponent, you multiply exponents. Also remember when you are multiplying numbers and variables and the whole thing is raised to an exponent, you can remove parentheses and "push through" the exponent. Example: $(2x^3y)^2=4x^6y^2$.
- **combine bases to combine exponents**. You should add exponents of common bases if you multiplying, and subtract exponents of common bases if you are dividing (you can subtract "up", or subtract "down", **starting with the largest exponent**, to get the positive exponent). Sometimes you have to match the bases first in order to combine exponents – see last example below. Examples: $a^2a^3 = a^5$

  $\frac{a^5}{a^3} = a^2$ $\frac{a^3}{a^5} = \frac{1}{a^2} = a^{-2}$ (which is a^{-2}).
- **get rid of negative exponents**. To get rid of negative exponents, you can simply move a negative exponent in the denominator to the numerator and make it positive, or vice versa. Examples:

  $a^{-4} = \frac{1}{a^4}$ $\left(\frac{a}{b}\right)^{-2} = \left(\frac{b}{a}\right)^2$.
- **simplify any numbers** (like $\sqrt{4} = 2$. Also, remember to simplify radicals by taking out any factors of perfect squares (under a square root), cubes (under a cube root), and so on. Example: $\sqrt{50x^2} = \sqrt{25 \cdot 2 \cdot x^2} = \sqrt{25} \cdot \sqrt{2} \cdot \sqrt{x^2} = 5x\sqrt{2}$. And most teachers will want you to [rationalize radical fractions].

Remember that, for the variables, we can divide the exponents inside by the root index – if it goes in exactly, we can take the variable to the outside; if there are any remainders, we have to leave the variables under the root sign. For example, $\sqrt[3]{x^5y^{12}} = x^1y^4\sqrt[3]{x^2} = xy^4\sqrt[3]{x^2}$ **,** since **5** divided by **3** is **1**, with **2** left over (for the *x*), and **12** divided by **3** is **4** (for the *y*). See how we could have just divided the exponents inside by the root outside, to end up with the rational (fractional) exponent (sort of like turning improper fractions into mixed fractions in the exponents): $\sqrt[3]{x^5y^{12}} = x^{\frac{5}{3}} \cdot y^{\frac{12}{3}} = x^{\frac{3}{3}} \cdot x^{\frac{2}{3}} \cdot y^4 = x \cdot x^{\frac{2}{3}} \cdot y^4 = xy^4\sqrt[3]{x^2}$?

- **combine any like terms**. If you're adding or subtracting terms with the same numbers (coefficients) and/or variables, you can put these together. Almost think of a **radical expression** (like 2–√) **like another variable**. Example: $4x^2\sqrt{2} - 2x^2\sqrt{2} = 2x^2\sqrt{2}$.

Now let us put it altogether. Here are some (difficult) examples. Just remember that you have to be really, really careful doing these!