If we examine the positions of the vertices (maxima) of a series of trajectories, all launched from a zero initial height, with the same initial velocity, an interesting pattern emerges. Consider the expressions for the vertex coordinates:

\[ x_V(\theta) := \frac{v_0^2}{g} \sin(\theta) \cos(\theta) \]
\[ y_V(\theta) := \frac{v_0^2}{2g} \sin(\theta)^2 \]

These parametric equations appear to be defining an ellipse, when \( \theta \) ranges from zero to 90 degrees. We can convert these expressions to a single \( y(x) \) function and examine that result to see if it is in fact an ellipse. First we use some trig identities to write

\[ x_V(\theta) = \frac{v_0^2}{2g} \sin(2\theta) \]
\[ y_V(\theta) = \frac{v_0^2}{4g} (1 - \cos(2\theta)) \]

Using the parametric format first, we have from analytic geometry that an ellipse is defined by

\[ x(\tau) = a \cos(\tau) \quad y(\tau) = b \sin(\tau) \]

If we define \( a := \frac{v_0^2}{2g} \) and \( \tau = 2\theta \) then

\[ x_V(\tau) = a \sin(\tau) \quad y_V(\tau) = -\frac{a}{2} \cos(\tau) + \frac{a}{2} \]

But it is also the case that

\[ x_V(\tau) = a \cos(\frac{\pi}{2} - \tau) \quad y_V(\tau) = \frac{a}{2} - \frac{a}{2} \sin(\frac{\pi}{2} - \tau) \]

and further, reversing the sign of the arguments (which must be the same) to get rid of the minus in \( y \),

\[ x_V(\tau) = a \cos(\tau - \frac{\pi}{2}) \quad y_V(\tau) = \frac{a}{2} + \frac{a}{2} \sin(\tau - \frac{\pi}{2}) \]

since \( \sin(-z) = -\sin(z) \) and \( \cos(-z) = \cos(z) \). Now we see that the argument is just some new parameter \( \phi \), so that we have the ellipse format, with center at \( (0, a/2) \), and minor axis in the \( y \)-direction:

\[ x_V(\phi) = a \cos(\phi) \quad y_V(\phi) = \frac{a}{2} + \frac{a}{2} \sin(\phi) \]

The position of the center of the ellipse is indicated by a square in the figure above. Note that \( \phi \) is measured from this center, and sweeps from negative 90 degrees to positive 90 degrees.
For the cartesian form, we return to Eq(1), solve for the trig functions and then square:

\[
\sin(2 \theta)^2 = \frac{4 x v^2 g^2}{v_0^4} \quad \cos(2 \theta)^2 = \left(1 - \frac{4 g y v}{v_0^2}\right)^2
\]

Since \( \sin(2 \theta)^2 + \cos(2 \theta)^2 = 1 \) then

\[
4 x v^2 g^2 + \left(\frac{v_0^2}{4} - 4 g y v\right)^2 = v_0^4
\]

\[
x^2 + \left(0 - 2 v_0^2 \frac{y v}{g} + 4 y^2\right) = 0 \quad 4 y^2 - 2 \frac{v_0^2}{g} y + x^2
\]

\[
y^2 - \frac{v_0^2}{2 g} y + \frac{v_0^4}{16 g^2} - \frac{x^2}{4} = 0
\]

complete square

\[
\left(y - \frac{v_0^2}{4 g}\right)^2 + \frac{x^2}{4} = \frac{v_0^4}{16 g^2}
\]

which is, again, the equation of an ellipse, with center at \((0, a/2)\) and minor axis 1/2 the major axis

This figure shows a few example trajectories along with the ellipse defined above. Clearly the vertices fall along this ellipse.