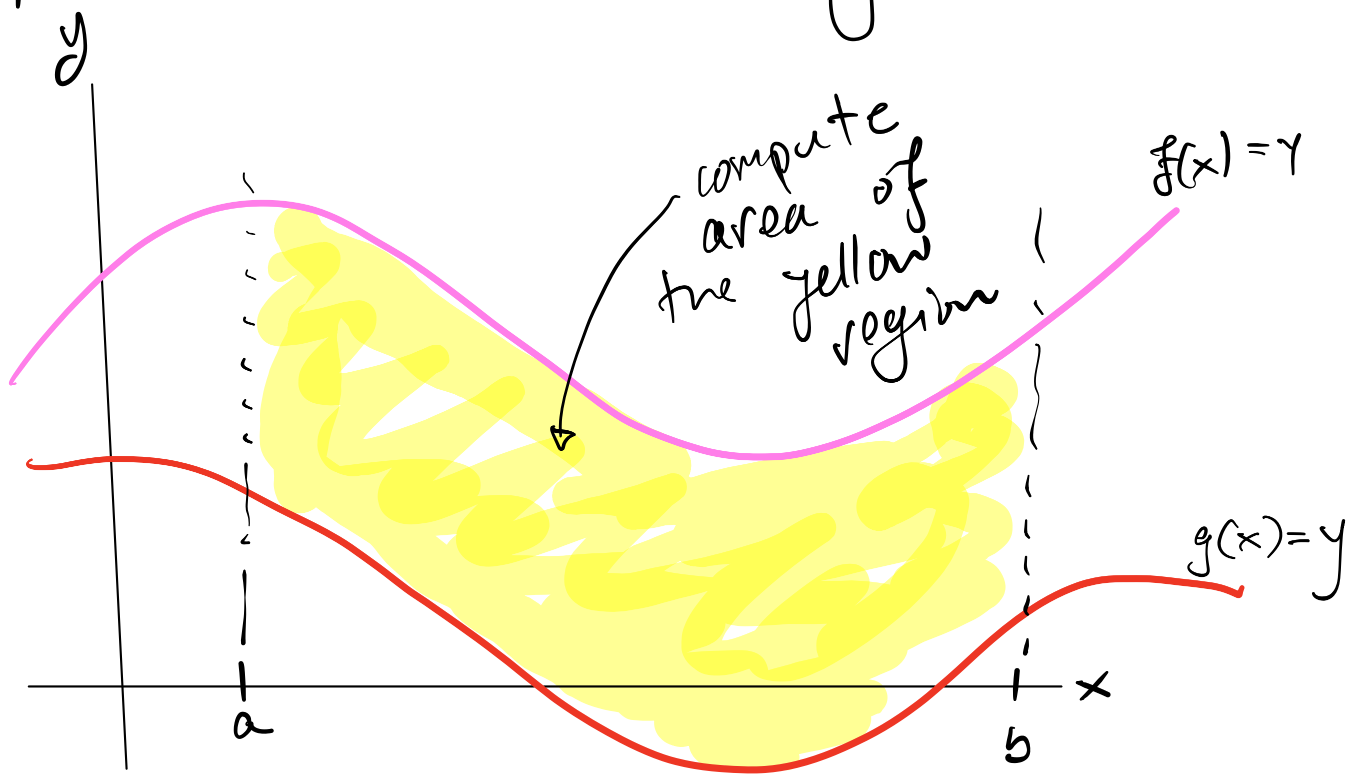
Class 3 §6.1

Applications of Integration



"Standard procedure"

"partition and sum"

(i) subdivide the interval $[a,b]$ into n strips of width $\Delta x = \frac{b-a}{n}$

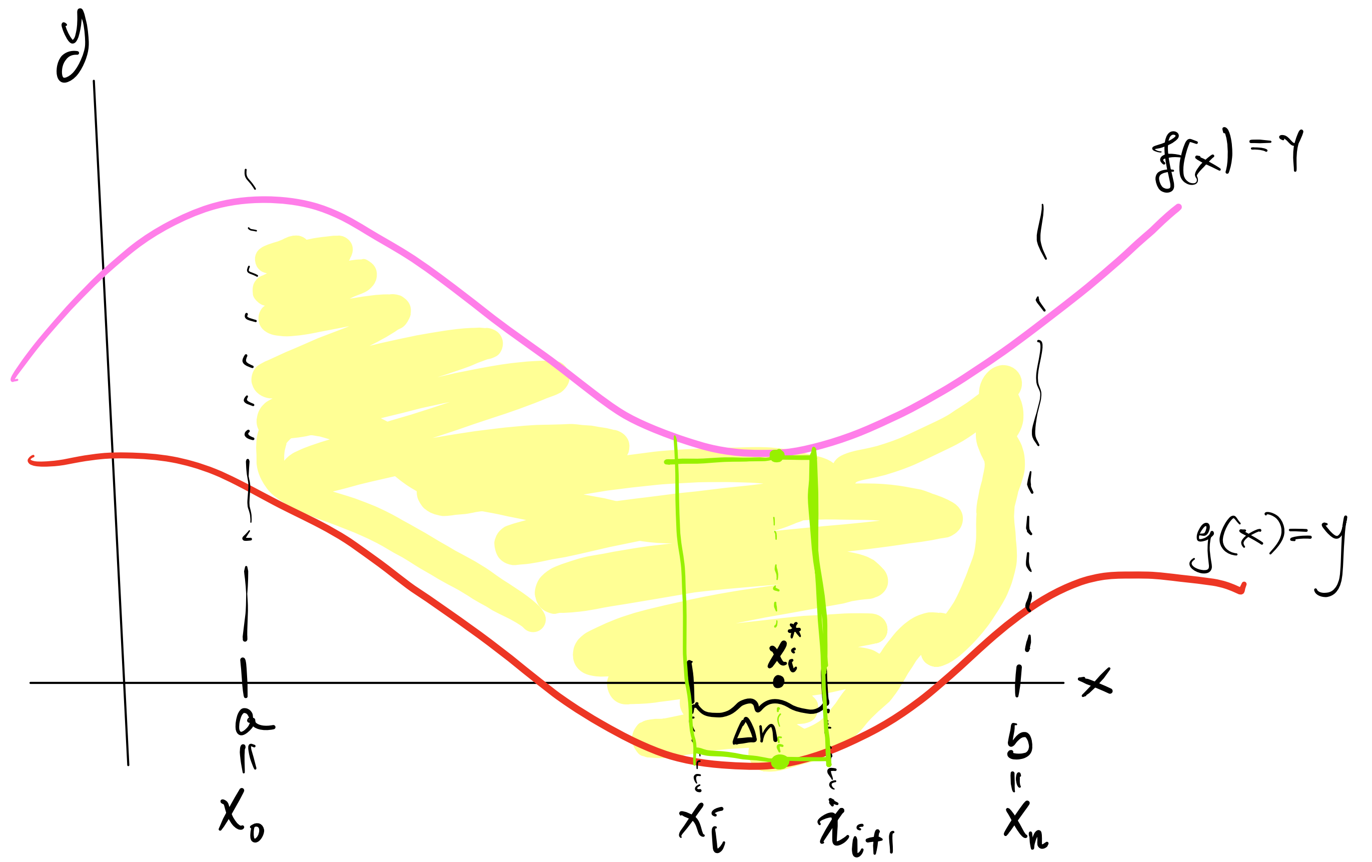
(ii) pick a point x_i^* in $[x_i, x_{i+1}]$

e.g. x_i, x_{i+1}, $\frac{x_{i+1}-x_i}{2}$

left right mid

(iii) Construct a rectangle of width Δx and height $f(x_i^*) - g(x_i^*)$

(iv) Add up the areas $\sum_{i=1}^{n} \big(f(x_i^*) - g(x_i^*)\big) \cdot \Delta x$



(v) Let $n \longrightarrow \infty$ (or $\Delta x \longrightarrow 0$)

$$\sum_{i=1}^{n}\left(f(x_i^*)-g(x_i^*)\right)\cdot\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x - \sum_{i=1}^{n} g(x_i^*)\Delta x$$

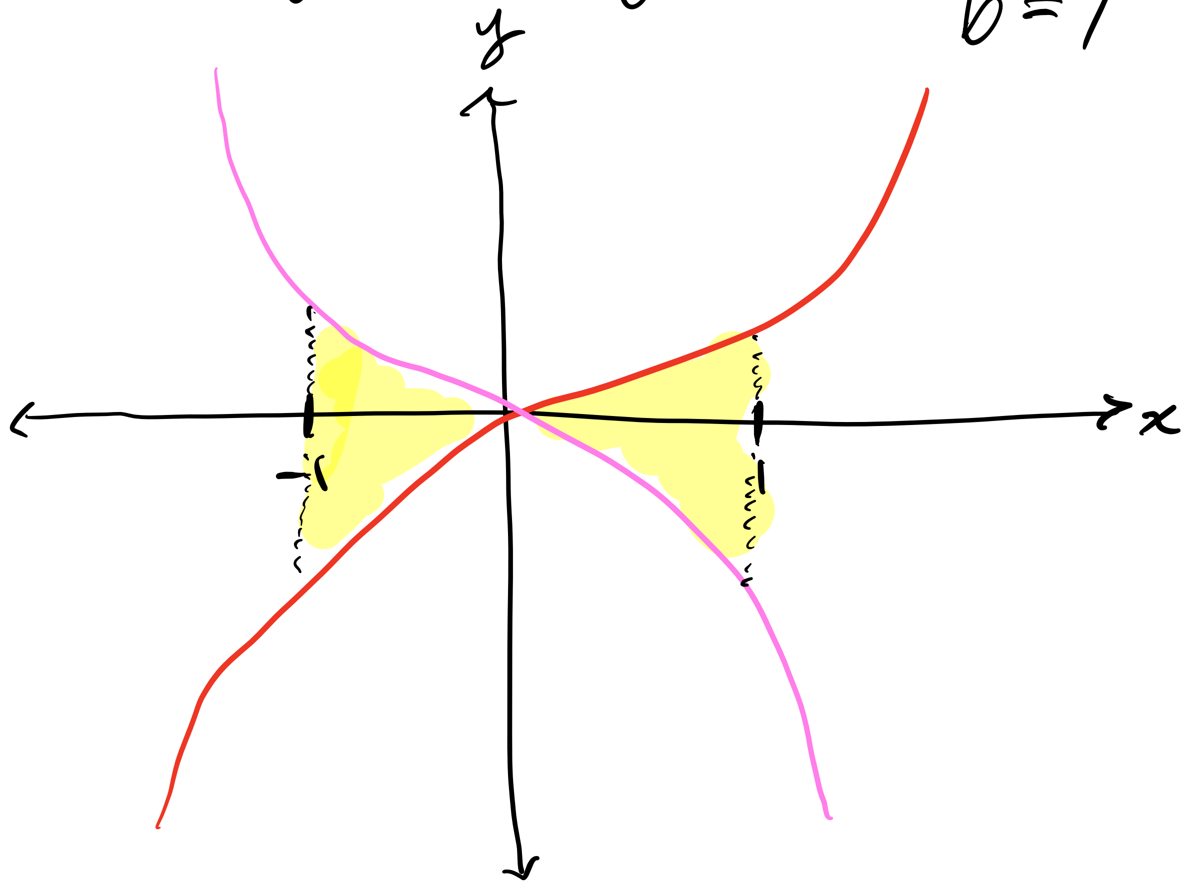
as $n \to \infty$ $\rightsquigarrow$ $\displaystyle\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$= \int_a^b f(x)-g(x)\,dx$$

<u>Punchline</u>: If f and g are continuous functions on $[a,b]$ and $f(x) \geq g(x)$ for every $a \leq x \leq b$ then the area between f and g as x goes from a to b is:

$$\int_a^b f(x)-g(x)\,dx$$

Ex: Let $f = x^3$, $g = -x^3$, $a = -1$, $b = 1$



$$\int_{-1}^{1} x^3 - (-x^3)\,dx = \int_{-1}^{1} 2x^3\,dx$$

$$= \frac{2}{4} x^4 \Big|_{-1}^{1}$$

$$= \frac{2}{4} \cdot 1^4 - \frac{2}{4}(-1)^4$$

$$= \frac{1}{2} - \frac{1}{2} = 0 \quad ???$$

OTOH: Just compute $\int_0^1 x^3-(-x^3)\,dx$, then multiply by 2:

$$\int_0^1 2x^3 = \frac{1}{2}x^4\Big|_0^1 = \frac{1}{2}$$

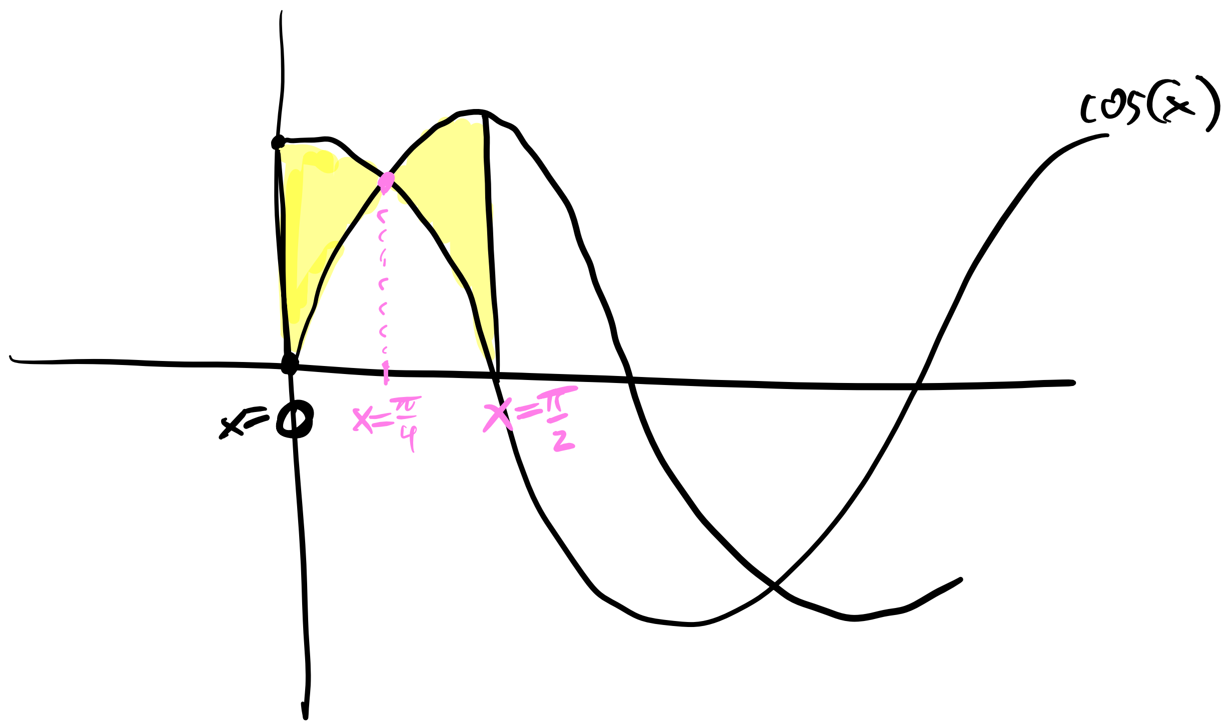
$\Longrightarrow$ Suggests the actual area is $\frac{1}{2}+\frac{1}{2}=1$.

Punchline 2: To compute the area between two functions f and g as $a \leq x \leq b$, we compute

$$\int_a^b |f-g|\,dx$$

absolute values are not really "formulaically" integrable

Ex: Find the area between $y=\sin(x)$ and $y=\cos(x)$ as x goes from 0 to $\pi/2$



$\sin(0)=0$ $\quad$ $\sin(\frac{\pi}{2})=1$

$\cos(0)=1$ $\quad$ $\cos(\pi/2)=0$

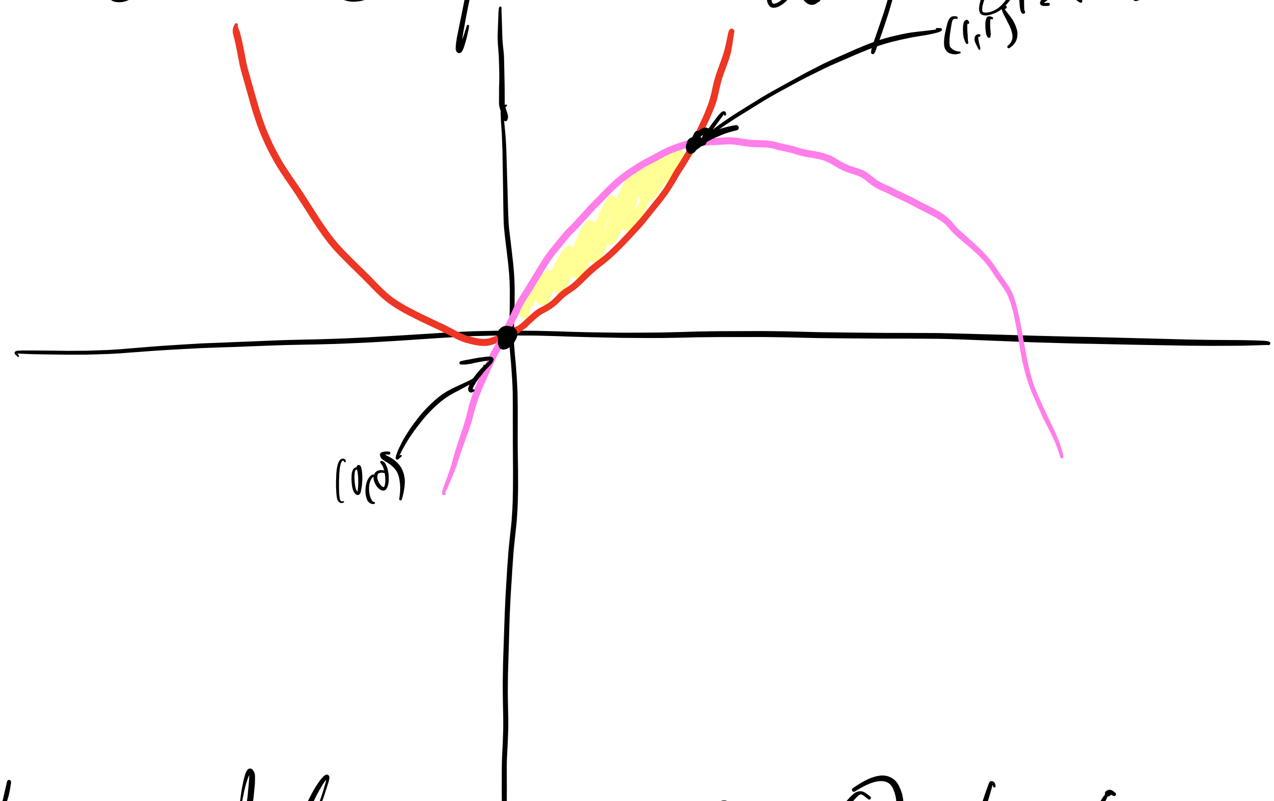
$$\sin(x)=\cos(x) \Rightarrow x=\frac{\pi}{4}$$

So break into two pieces where sin "passes" cos, i.e

$$\int_0^{\pi/4} \cos(x)-\sin(x)\,dx + \int_{\pi/4}^{\pi/2} \sin(x)-\cos(x)\,dx$$

$$= \text{AREA} = 2\sqrt{2}-2$$

Ex: Find the area of the region bounded by the curves $y=x^2$ and $y=2x-x^2$



You determine $a=0,\ b=1$

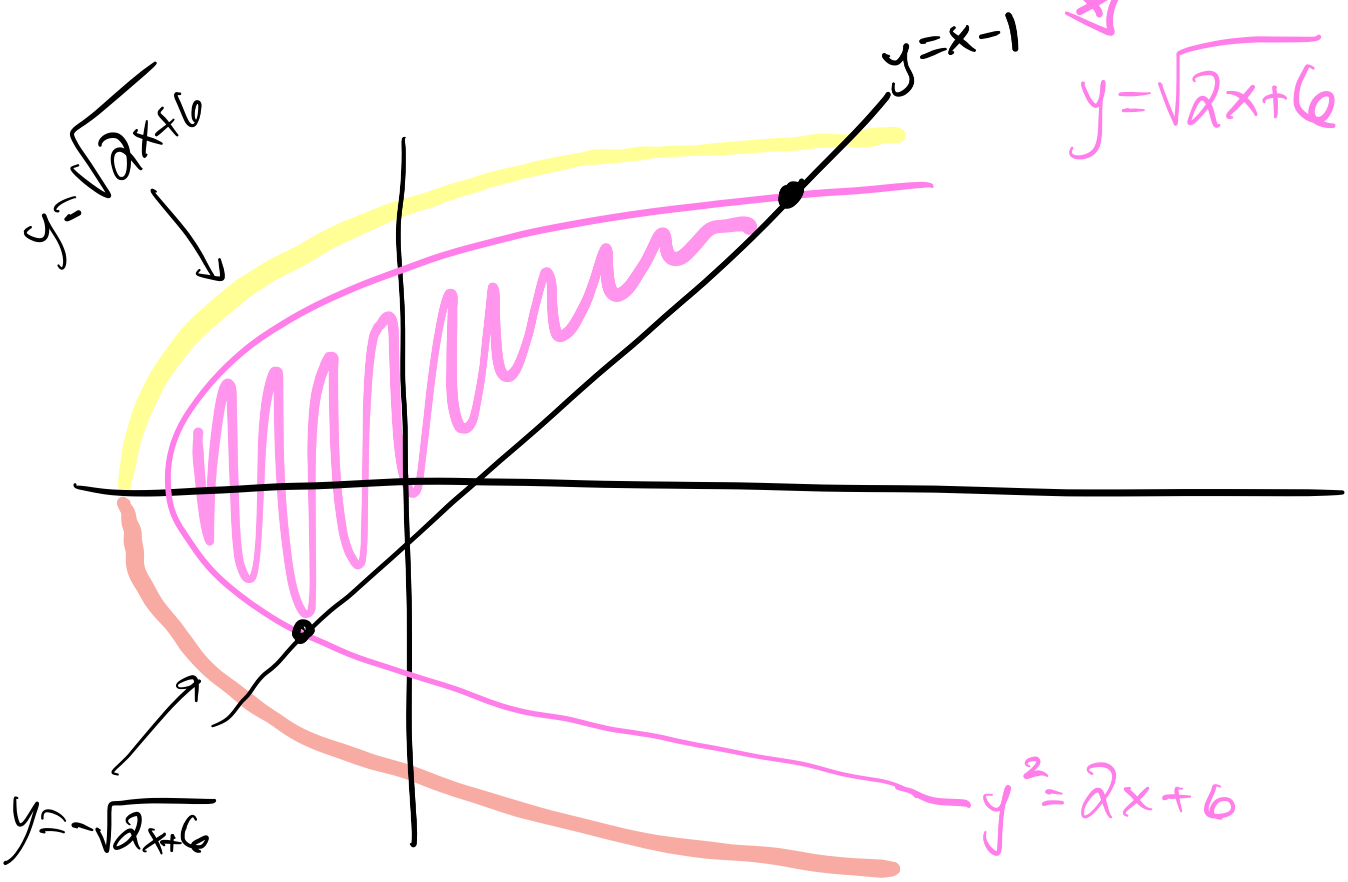
So area is:

$$\int_0^1 (2x-x^2)-x^2\,dx$$

$$=\int_0^1 2x-2x^2\,dx=\frac{1}{3}$$

Ex: Find the area of the region bounded by the curves

$y=x-1$ $\qquad$ $y^2=2x+6$ $\rightarrow$ $y=\sqrt{2x+6}$



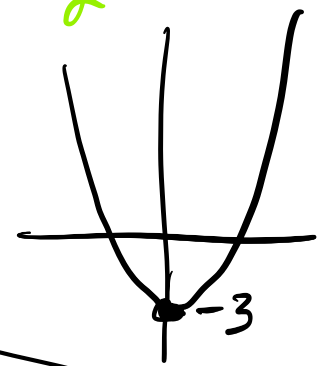
ASIDE:

$$y^2 = 2x + 6$$

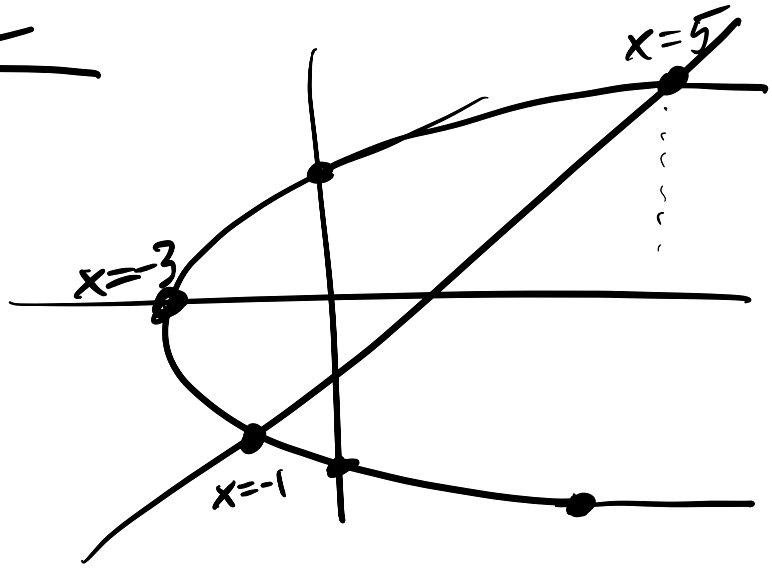
$$\Rightarrow y^2 - 6 = 2x$$

$$\Rightarrow \frac{y^2}{2} - 3 = x$$

$$y = \frac{x^2}{2} - 3$$



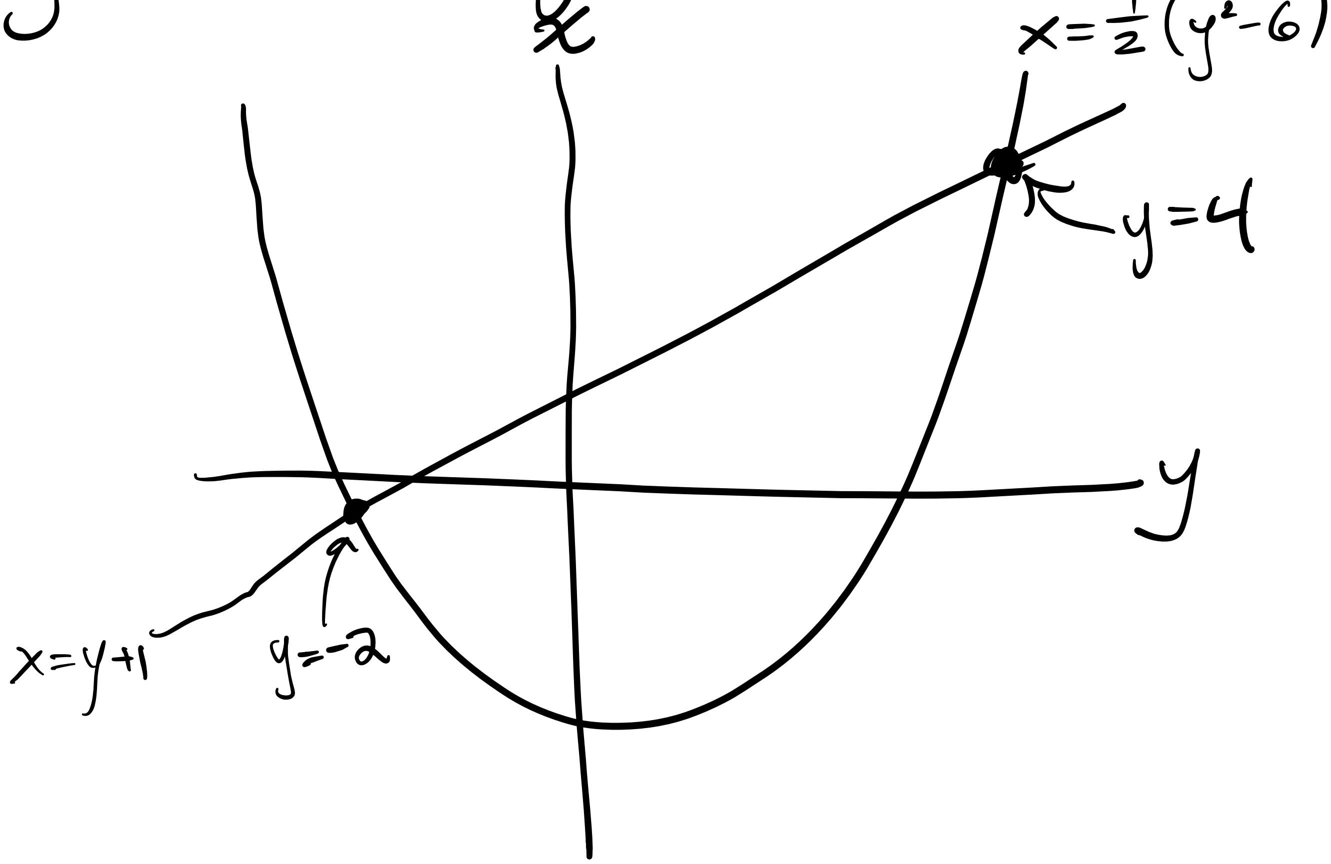
OPTION I



$$\int_{-3}^{-1} \sqrt{2x+6} - \left(-\sqrt{2x+6}\right) dx$$

$$+ \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx$$

OPTION II Integrate w/r/t y instead of x i.e.



Compute:

$$\int_{-2}^{4} (y+1) - \left(\frac{1}{2}(y^2-6)\right) dy$$

$$= \int_{-2}^{4} y + 1 - \frac{y^2}{2} + 3 \, dy$$

$$= \cdots\cdots = 18$$