**EXERCISES [MAI 1.16]**

**EIGENVALUES - EIGENVECTORS**

**SOLUTIONS**

**Compiled by: Christos Nikolaidis**

## A. Paper 1 questions (SHORT)

**1.** (a) det($\boldsymbol{A} - \lambda \boldsymbol{I}$) = $\begin{vmatrix} 3-\lambda & -2 \\ -3 & 4-\lambda \end{vmatrix}$ = $(3-\lambda)(4-\lambda) - 6 = \lambda^2 - 7\lambda + 6$

(b) $\lambda^2 - 7\lambda + 6 = 0 \Leftrightarrow \lambda = 1$ or $\lambda = 6$

(c) For $\lambda = 1$ we obtain the system

$$\begin{aligned} 2x - 2y &= 0 \\ -3x + 3y &= 0 \end{aligned} \Rightarrow y = x \Rightarrow \frac{x}{y} = 1.$$ Eigenvector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any multiple)

For $\lambda = 6$ we obtain the system

$$\begin{aligned} -3x - 2y &= 0 \\ -3x - 2y &= 0 \end{aligned} \Rightarrow 2y = -3x \Rightarrow \frac{x}{y} = -\frac{2}{3}.$$ Eigenvector: $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ (or any multiple)

**Notice: the systems can be solved by GDC.**

**2.** (a) $\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)(4-\lambda) - 3 = 0 \Leftrightarrow \lambda^2 - 6\lambda + 5 = 0$

$\lambda = 1$, $\lambda = 5$

(b) For $\lambda = 1$ we obtain the system

$$\begin{aligned} x + 3y &= 0 \\ x + 3y &= 0 \end{aligned} \Rightarrow x = -3y \Rightarrow \frac{x}{y} = -3.$$ Eigenvector: $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (or any multiple)

For $\lambda = 5$ we obtain the system

$$\begin{aligned} -3x + 3y &= 0 \\ x - y &= 0 \end{aligned} \Rightarrow y = x \Rightarrow \frac{x}{y} = 1.$$ Eigenvector: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (or any multiple)

**3.** (a) $\begin{vmatrix} 2-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)(4-\lambda) = 0$

$\lambda = 2$, $\lambda = 4$

(b) For $\lambda = 2$ we obtain the system

$$\begin{aligned} 0x + 0y &= 0 \\ x + 2y &= 0 \end{aligned} \Rightarrow x = -2y \Rightarrow \frac{x}{y} = -2.$$ Eigenvector: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ (or any multiple)

For $\lambda = 4$ we obtain the system

$$\begin{aligned} -2x + 0y &= 0 \\ x + 0y &= 0 \end{aligned} \Rightarrow x = 0$$ ($y$ free variable). Eigenvector: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (or any multiple)

**4.** (a) $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow (1-\lambda)^2 - 1 = 0 \Leftrightarrow \lambda^2 - 2\lambda = 0$

$\lambda = 0$, $\lambda = 2$

(b) For $\lambda = 0$ we obtain the system

$$\begin{matrix} x+y=0 \\ x+y=0 \end{matrix} \Rightarrow y=-x \Rightarrow \frac{x}{y} = -1. \qquad \text{Eigenvector: } \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ (or any multiple)}$$

For $\lambda = 2$ we obtain the system

$$\begin{matrix} -x+y=0 \\ x-y=0 \end{matrix} \Rightarrow y=x \Rightarrow \frac{x}{y} = 1. \text{ Eigenvector: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (or any multiple)}$$

(c) $D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ (**OR** $D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ )

(d) $\boldsymbol{D = P^{-1}MP}$

**5.** (a) $\begin{vmatrix} 2-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)^2 + 3 = 0$ No real roots

(b) $\begin{vmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Leftrightarrow (2-\lambda)^2 - 3 = 0 \Leftrightarrow \lambda^2 - 4\lambda + 1 = 0$

$\lambda = 2 \pm \sqrt{3}$

(b) For $\lambda = 2+\sqrt{3}$ we obtain the system

$$\begin{matrix} -\sqrt{3}x+y=0 \\ 3x-\sqrt{3}y=0 \end{matrix} \Rightarrow y=\sqrt{3}x \Rightarrow \frac{x}{y} = \frac{1}{\sqrt{3}}. \text{ Eigenvector: } \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \text{ (or any multiple)}$$

For $\lambda = 2-\sqrt{3}$ we obtain the system

$$\begin{matrix} \sqrt{3}x+y=0 \\ 3x+\sqrt{3}y=0 \end{matrix} \Rightarrow y=-\sqrt{3}x \Rightarrow \frac{x}{y} = -\frac{1}{\sqrt{3}}. \ . \text{ Eigenvector: } \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \text{ (or any multiple)}$$

## B. Paper 2 questions (*LONG*)

**6.** (a) $\boldsymbol{M}^2 = \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix}$ $\quad 6\boldsymbol{M} = \begin{pmatrix} 12 & -6 \\ -18 & 24 \end{pmatrix}$

$$\boldsymbol{M}^2 - 6\boldsymbol{M} + k\boldsymbol{I} = \boldsymbol{O} \Leftrightarrow \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Leftrightarrow k = 5$$

(b) $\begin{vmatrix} 2-\lambda & -1 \\ -3 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5$

(c) the matrix satisfies the characteristic polynomial

(d) $\lambda = 1$, $\lambda = 5$

(e) For λ = 1 we obtain the system

$$\begin{matrix} x - y = 0 \\ -3x + 3y = 0 \end{matrix} \Rightarrow y = x \Rightarrow \frac{x}{y} = 1. \text{ Eigenvector: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (or any multiple)}$$

For λ = 5 we obtain the system

$$\begin{matrix} -3x - y = 0 \\ -3x - y = 0 \end{matrix} \Rightarrow y = -3x \Rightarrow \frac{x}{y} = -\frac{1}{3}. \text{ Eigenvector: } \begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ (or any multiple)}$$

**7.** (a) $\boldsymbol{A} = \boldsymbol{PDP}^{-1} \Rightarrow \boldsymbol{A}^2 = \boldsymbol{PDP}^{-1}\,\boldsymbol{PDP}^{-1} = \boldsymbol{PDIDP}^{-1} = \boldsymbol{PDDP}^{-1} = \boldsymbol{PD}^2\boldsymbol{P}^{-1}$

(b) $\begin{vmatrix} 3-\lambda & 2 \\ a & 6-\lambda \end{vmatrix} = (3-\lambda)(6-\lambda) - 2a = \lambda^2 - 9\lambda + 18 - 2a = 0$

For $\lambda = 4$, $16 - 36 + 18 - 2a = 0 \Leftrightarrow -2 - 2a = 0 \Leftrightarrow a = -1$

(c) $\lambda^2 - 9\lambda + 20 = 0 \Leftrightarrow \lambda = 4$, $\lambda = 5$ so the other eigenvalue is $\lambda = 5$

(d) For λ = 4 we obtain the system

$$\begin{matrix} -x + 2y = 0 \\ -x + 2y = 0 \end{matrix} \Rightarrow x = 2y \Rightarrow \frac{x}{y} = 2. \text{ Eigenvector: } \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ (or any multiple)}$$

For λ = 5 we obtain the system

$$\begin{matrix} -2x + 2y = 0 \\ -x + y = 0 \end{matrix} \Rightarrow y = x \Rightarrow \frac{x}{y} = 1. \text{ Eigenvector: } \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ (or any multiple)}$$

(e) $\boldsymbol{D} = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

(f) $\boldsymbol{A}^n = \boldsymbol{PD}^n\boldsymbol{P}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4^n & -4^n \\ -5^n & 2\times 5^n \end{pmatrix}$

$$= \begin{pmatrix} 2\times 4^n - 5^n & -2\times 4^n + 2\times 5^n \\ 4^n - 5^n & -4^n + 2\times 5^n \end{pmatrix} = \begin{pmatrix} 2\times 4^n - 5^n & 2\times 5^n - 2\times 4^n \\ 4^n - 5^n & 2\times 5^n - 4^n \end{pmatrix}$$

**8.** (a) $\begin{vmatrix} 0.8-\lambda & 0.1 \\ 0.2 & 0.9-\lambda \end{vmatrix} = (0.8-\lambda)(0.9-\lambda)-0.02 = \lambda^2 - 1.7\lambda + 0.7 = 0$

$\Leftrightarrow \lambda = 1,\ \lambda = 0.7$

For $\lambda = 1$ we obtain the system

$$\begin{aligned} -0.2x+0.1y&=0 \\ 0.2x-0.1y&=0 \end{aligned} \Rightarrow 0.2x = 0.1y \ \Rightarrow \frac{x}{y} = \frac{1}{2}.$$ Eigenvector: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (or any multiple)

For $\lambda = 0.7$ we obtain the system

$$\begin{aligned} 0.1x+0.1y&=0 \\ 0.2x+0.2y&=0 \end{aligned} \Rightarrow 0.1x = -0.1y \ \Rightarrow \frac{x}{y} = -1.$$ Eigenvector: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (or any multiple)

(b) $\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$

$$\boldsymbol{M}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1^n & 0 \\ 0 & 0.7^n \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2\times 0.7^n & 0.7^n \end{pmatrix}$$

$$\frac{1}{3}\begin{pmatrix} 1+2\times 0.7^n & 1-0.7^n \\ 2-2\times 0.7^n & 2+0.7^n \end{pmatrix}$$

(c) As $n$ tends to infinity, $0.7^n$ tends to 0 and thus $\boldsymbol{M}^n$ tends to $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix}$