[MAA 5.23] MACLAURIN SERIES – EXTENSION OF BINOMIAL THEOREM

SOLUTIONS

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O. Practice questions

1. (a)

$f(x)=e^{2x}$ $\quad f(0)=1$

$f'(x)=2e^{2x}$ $\quad f'(0)=2$

$f''(x)=4e^{2x}$ $\quad f''(0)=4$

$f'''(x)=8e^{2x}$ $\quad f'''(0)=8$

$f^{(4)}(x)=16e^{2x}$ $\quad f^{(4)}(0)=16$

$$f(x)=1+2x+\frac{4}{2!}x^2+\frac{8}{3!}x^3+\frac{16}{4!}x^4\cdots=1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4\cdots$$

(b) $$e^{2x}=1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}+\cdots=1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4\cdots$$

2. (a)

$f(x)=(2+x)^3$ $\quad f(0)=8$

$f'(x)=3(2+x)^2$ $\quad f'(0)=12$

$f''(x)=6(2+x)$ $\quad f''(0)=12$

$f'''(x)=6$ $\quad f'''(0)=6$

$$f(x)=8+12x+\frac{12}{2!}x^2+\frac{6}{3!}x^3+0+\cdots=8+12x+6x^2+x^3$$

(b) $$(2+x)^3=2^3+3\times2^2\times x+3\times2\times x^2+x^3=8+12x+6x^2+x^3$$

3. (a)

$f(x)=\dfrac{1}{(2+x)^2}=(2+x)^{-2}$ $\quad f(0)=\dfrac{1}{4}$

$f'(x)=-2(2+x)^{-3}=\dfrac{-2}{(2+x)^3}$ $\quad f'(0)=-\dfrac{1}{4}$

$f''(x)=6(2+x)^{-4}=\dfrac{6}{(2+x)^4}$ $\quad f''(0)=\dfrac{3}{8}$

$$f(x)=\frac{1}{4}-\frac{1}{4}x+\frac{3/8}{2!}x^2+\cdots=\frac{1}{4}-\frac{1}{4}x+\frac{3}{16}x^2+\cdots$$

(b) $$(2+x)^{-2}=2^{-2}\left(1+\frac{x}{2}\right)^{-2}=\frac{1}{4}\left(1+(-2)\frac{x}{2}+\frac{(-2)(-3)}{2}\left(\frac{x}{2}\right)^2+\cdots\right)=\frac{1}{4}-\frac{1}{4}x+\frac{3}{16}x^2+\cdots$$

(c) The series converges when $\left|\frac{x}{2}\right|<1\Leftrightarrow|x|<2\Leftrightarrow-2<x<2$

4. (a) $f(x)=\sqrt{2+x}$ $\qquad f(0)=\sqrt{2}$

$f'(x)=\dfrac{1}{2\sqrt{2+x}}=\dfrac{1}{2}(2+x)^{-\frac{1}{2}}$ $\qquad f'(0)=\dfrac{1}{2\sqrt{2}}$

$f''(x)=-\dfrac{1}{4}(2+x)^{-\frac{3}{2}}=-\dfrac{1}{4\sqrt{(2+x)^3}}$ $\qquad f''(0)=-\dfrac{1}{8\sqrt{2}}$

The Maclaurin series is

$$f(x)=\sqrt{2}+\frac{1}{2\sqrt{2}}x-\frac{1}{16\sqrt{2}}x^2+\cdots=\sqrt{2}+\frac{\sqrt{2}}{4}x-\frac{\sqrt{2}}{32}x^2+\cdots$$

(b) $$(2+x)^{\frac{1}{2}}=2^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}=\sqrt{2}\left(1+\frac{1}{2}\frac{x}{2}+\frac{\frac{1}{2}(-\frac{1}{2})}{2}\left(\frac{x}{2}\right)^2+\cdots\right)$$

$$=\sqrt{2}\left(1+\frac{x}{4}+\frac{x^2}{32}+\cdots\right)=\sqrt{2}+\frac{\sqrt{2}}{4}x-\frac{\sqrt{2}}{32}x^2+\cdots$$

5. (a) $$f(x)=\sqrt{x^2+1}=\left(1+x^2\right)^{\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{\frac{1}{2}(-\frac{1}{2})}{2}x^4+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^6+\cdots$$

$$=1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+\cdots$$

(b) The Maclaurin series of $f'(x)=\dfrac{x}{\sqrt{x^2+1}}$ is

$$\left(1+\frac{1}{2}x^2-\frac{1}{8}x^4+\frac{1}{16}x^6+\cdots\right)'=1+x-\frac{1}{2}x^3+\frac{3}{8}x^5+\cdots$$

6. By appropriate substitutions of x on $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots$ we obtain

(a) Substitute $-x$ for x :

$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots=1-x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots$$

(b) Substitute x^2 for x on (a):

$$e^{-x^2}=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots=1-x^2+\frac{x^4}{2}-\frac{x^6}{6}+\cdots$$

(c) Multiply x by the expansion in (a):

$$xe^x=x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)=x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\cdots=x+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\cdots$$

(d) Multiply x^2 by the expansion in (a):

$$x^2e^{-x}=x^2\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\right)=x^2-x^3+\frac{x^4}{2!}-\frac{x^5}{3!}+\cdots=x^2-x^3+\frac{x^4}{2}-\frac{x^5}{6}+\cdots$$

(e) Subtract (a) from the original:

$$e^x - e^{-x} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = 2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + \cdots$$

$$= 2x + \frac{x^3}{3} + \frac{x^5}{60} + \cdots$$

(f) Substitute $4x$ for x on e^x and then multiply by $(x+1)$

$$(x+1)e^{4x} = (x+1)\left(1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \cdots\right)$$

$$= x\left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \cdots\right) + \left(1 + 4x + 8x^2 + \frac{32x^3}{3} + \cdots\right)$$

$$= 1 + 5x + 12x^2 + \frac{56x^3}{3} + \cdots$$

A. Exam style questions (SHORT)

7. (a) **METHOD 1**

$f(x) = \ln(1 + e^x)$; $f(0) = \ln 2$

$f'(x) = \dfrac{e^x}{1+e^x}$; $f'(0) = \dfrac{1}{2}$

$f''(x) = \dfrac{e^x(1+e^x) - 2e^{2x}}{(1+e^x)^2}$; $f''(0) = \dfrac{1}{4}$

$\ln(1 + e^x) = \ln 2 + \dfrac{1}{2}x + \dfrac{1}{8}x^2 + ...$

METHOD 2

$$\ln(1 + e^x) = \ln(1 + 1 + x + \frac{1}{2}x^2 + ...)$$

$$= \ln 2 + \ln(1 + \frac{1}{2}x + \frac{1}{4}x^2 + ...)$$

$$= \ln 2 + \left(\frac{1}{2}x + \frac{1}{4}x^2 + ...\right) - \frac{1}{2}\left(\frac{1}{2}x + \frac{1}{4}x^2 + ...\right)^2 + ...$$

$$= \ln 2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2 + ...$$

$$= \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + ...$$

(b) **METHOD 1**

$$\lim_{x\to 0}\frac{2\ln(1+e^x) - x - \ln 4}{x^2} = \lim_{x\to 0}\frac{\left(2\ln 2 + x + \frac{x^2}{4} + \cdots\right) - x - \ln 4}{x^2} = \lim_{x\to 0}\left(\frac{1}{4} + \text{powers of } x\right) = \frac{1}{4}$$

METHOD 2 using l'Hôpital's Rule

$$\lim_{x\to 0}\frac{2\ln(1+e^x) - x - \ln 4}{x^2} = \lim_{x\to 0}\frac{2e^x \div (1+e^x) - 1}{2x} = \lim_{x\to 0}\frac{2e^x \div (1+e^x)^2}{2} = \frac{1}{4}$$

8. (a) Constant term = 0

(b) $f'(x)=\dfrac{1}{1-x}$

$f''(x)=\dfrac{1}{(1-x)^2}$

$f'''(x)=\dfrac{2}{(1-x)^3}$

$f'(0) = 1; f''(0) = 1; f'''(0) = 2$

$f(x)=0+\dfrac{1\times x}{1!}+\dfrac{1\times x^2}{2!}+\dfrac{2\times x^3}{3!}+\ldots= x+\dfrac{x^2}{2}+\dfrac{x^3}{2}$

(c) $\dfrac{1}{1-x}=2\Rightarrow x=\dfrac{1}{2}$

$\ln 2\approx\dfrac{1}{2}+\dfrac{1}{8}+\dfrac{1}{24}=\dfrac{2}{3}\ (0.667)$

9. (a) $f'(x)=\dfrac{\cos x}{1+\sin x}$

$f''(x)=\dfrac{-\sin x(1+\sin x)-\cos^2 x}{(1+\sin x)^2}=\dfrac{-\sin x-1}{(1+\sin x)^2}=\dfrac{-1}{1+\sin x}$

(b) $f'''(x)=\dfrac{\cos x}{(1+\sin x)^2}$

$f^{iv}(x)=\dfrac{-\sin x(1+\sin x)^2-2(1+\sin x)\cos^2 x}{(1+\sin x)^4}$

(c) $f(0) = 0,\ f'(0) = 1,\ f''(0) = -1,\ f'''(0) = 1,\ f^{\text{iv}}(0) = -2$

$\ln(1 + \sin x) = x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3-\dfrac{1}{12}x^4+\ldots$

10. (a) from $\dfrac{dy}{dx} = y\tan x + \cos x$,

$f'(0) = 1$

now $\dfrac{d^2y}{dx^2}= y\sec^2 x+\dfrac{dy}{dx}\tan x-\sin x$

$\Rightarrow f''(0)=-\dfrac{\pi}{2}$

Hence the Maclaurin series for y is

$y=-\dfrac{\pi}{2}+x-\dfrac{\pi x^2}{4}$

(b) When $x = 0.1$, $y\cong-\dfrac{\pi}{2}+0.1-\dfrac{\pi(0.1)^2}{4}=-\dfrac{\pi}{2}+0.1-\dfrac{\pi}{400}=0.1-\dfrac{201\pi}{400}$

11. (a) $f(x)=\dfrac{1}{3x+5}=(5+3x)^{-1}=5^{-1}(1+\dfrac{3}{5}x)^{-1}$

Hence

$$f(x)=\frac{1}{5}\left(1+(-1)\frac{3}{5}x+\frac{(-1)(-2)}{2!}\left(\frac{3}{5}x\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{3}{5}x\right)^3+\cdots\right)$$

$$=\frac{1}{5}\left(1+-\frac{3}{5}x+\frac{9}{25}x^2-\frac{27}{125}x^3+\cdots\right)$$

$$=\frac{1}{5}-\frac{3}{25}x+\frac{9}{125}x^2-\frac{27}{625}x^3+\cdots$$

(b) The series converges when

$$\left|\frac{3x}{5}\right|<1\Leftrightarrow|x|<\frac{5}{3}\Leftrightarrow-\frac{5}{3}<x<\frac{5}{3}$$

11. (a) $f(x)=\sqrt{3x+5}=(5+3x)^{\frac{1}{2}}=5^{\frac{1}{2}}(1+\dfrac{3}{5}x)^{\frac{1}{2}}$

Hence

$$f(x)=\sqrt{5}\left(1+\left(\frac{1}{2}\right)\frac{3}{5}x+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(\frac{3}{5}x\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{3}{5}x\right)^3+\cdots\right)$$

$$=\sqrt{5}\left(1+\frac{3}{10}x+\frac{\left(-\frac{1}{4}\right)}{2}\left(\frac{3}{5}x\right)^2+\frac{\left(\frac{3}{8}\right)}{6}\left(\frac{3}{5}x\right)^3+\cdots\right)$$

$$=\sqrt{5}\left(1+\frac{3}{10}x-\frac{1}{8}\left(\frac{9}{25}x^2\right)+\frac{1}{16}\left(\frac{27}{125}x^3\right)+\cdots\right)$$

$$=\sqrt{5}\left(1+\frac{3}{10}x-\frac{9}{200}x^2+\frac{27}{2000}x^3+\cdots\right)$$

$$=\sqrt{5}+\frac{3\sqrt{5}}{10}x-\frac{9\sqrt{5}}{200}x^2+\frac{27\sqrt{5}}{2000}x^3+\cdots$$

(b) The series converges when

$$\left|\frac{3x}{5}\right|<1\Leftrightarrow|x|<\frac{5}{3}\Leftrightarrow-\frac{5}{3}<x<\frac{5}{3}$$

B. Exam style questions (LONG)

13. (a) (i) $\ln(1 - \sin x) = \ln(1 + \sin(-x)) = -x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$

(ii) $\ln(1 + \sin x) + \ln(1 - \sin x) = \ln(1 - \sin^2 x) = \ln \cos^2 x$

So $\ln \cos^2 x = -x^2 - \frac{1}{6}x^4 + \ldots$

$\ln \cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots$

(iii) $\ln \sec x = -\ln \cos x = \frac{x^2}{2} + \frac{x^4}{12} + \ldots$

(b) Differentiating, $\frac{d}{dx}(\ln \cos x) = \frac{1}{\cos x} \times (-\sin x) = -\tan x$

The derivative of the series is $-x - \frac{1}{3}x^3 - \ldots$

Hence

$\tan x = x + \frac{1}{3}x^3 + \ldots$

(c) (i) $\frac{\ln \sec x}{x\sqrt{x}} = \frac{\sqrt{x}}{2} + \frac{x^2\sqrt{x}}{12} + \ldots$

Limit = 0

(ii) $\frac{\tan(x^2)}{\ln \cos x} = \frac{x^2 + \frac{x^4}{3} + \ldots}{-\frac{x^2}{2} - \frac{x^4}{12} + \ldots}$

$= \frac{1 + \frac{x^4}{3} + \ldots}{-\frac{1}{2} - \frac{x^2}{12} + \ldots} \to -2$ as $x \to 0$

so $\lim_{x \to 0}\left(\frac{\tan(x^2)}{\ln \cos x}\right) = -2$

(d) $\ln(1 + \sin x) - \ln(1 - \sin x) = \ln\left(\frac{1 + \sin x}{1 - \sin x}\right) \approx 2x + \frac{x^3}{3}$

let $x = \frac{\pi}{6}$ then, $\ln\left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right) = \ln 3 \approx 2\left(\frac{\pi}{6}\right) + \frac{\left(\frac{\pi}{6}\right)^3}{3}$

$= \frac{\pi}{3}\left(1 + \frac{\pi^2}{216}\right)$

14. (a) $f(x) = \ln \cos x$

$$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$$

$$f''(x) = -\sec^2 x$$

$$f'''(x) = -2 \sec x \sec x \tan x$$

$$f^{\text{iv}}(x) = -2 \sec^2 x (\sec^2 x) - 2 \tan x (2 \sec^2 x \tan x)$$

$$= -2 \sec^4 x - 4 \sec^2 x \tan^2 x$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots$$

$f(0) = 0,$

$f'(0) = 0,$

$f''(0) = -1,$

$f'''(0) = 0,$

$f^{\text{iv}}(0) = -2,$

$$\ln(\cos x) \approx -\frac{x^2}{2!} - \frac{2x^4}{4!} = -\frac{x^2}{2} - \frac{x^4}{12}$$

(b) **EITHER** Taking $x = \frac{\pi}{3}$

$$\ln \frac{1}{2} \approx -\frac{\left(\frac{\pi}{3}\right)^2}{2!} - \frac{2\left(\frac{\pi}{3}\right)^4}{4!} \Rightarrow -\ln 2 \approx -\frac{\frac{\pi^2}{9}}{2!} - \frac{2\frac{\pi^4}{81}}{4!}$$

$$\ln 2 \approx \frac{\pi^2}{18} + \frac{\pi^4}{972} = \frac{\pi^2}{9}\left(\frac{1}{2} + \frac{\pi^2}{108}\right)$$

OR Taking $x = \frac{\pi}{4}$

$$\ln \frac{1}{\sqrt{2}} \approx -\frac{\left(\frac{\pi}{4}\right)^2}{2!} - \frac{2\left(\frac{\pi}{4}\right)^4}{4!}$$

$$-\frac{1}{2}\ln 2 \approx \frac{\frac{\pi^2}{16}}{2!} - \frac{2\frac{\pi^4}{256}}{4!}$$

$$\ln 2 \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536} = \frac{\pi^2}{8}\left(\frac{1}{2} + \frac{\pi^2}{192}\right)$$