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STUDENT VERSION

Atmospheric CO_2

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Abstract: Students are introduced to the concept of a bifurcation in a first-order ordinary differential equation (ODE) through a modeling scenario involving atmospheric CO_2. CO_2 is taken as a parameter and temperature is a function of time. For low values of CO_2, there are three equilibrium points: one stable corresponding to present day temperatures, one stable corresponding to much higher temperatures, and an unstable in between. Once CO_2 is raised above the bifurcation point, the low-stable and unstable equilibrium point annihilate one another (a saddle-node bifurcation), resulting in the system converging to the lone remaining high-stable equilibrium point.

Students will use both graphing and numerical techniques to find equilibrium points. Matlab software for numerical solution of the ODE is included in an attached file.

INTRODUCTION

Suppose that the average earth temperature as a function of the level of atmospheric CO_2 is modeled using the following first-order ordinary differential equation (ODE):

$$\frac{dT}{dt} = -0.005T^3 + 0.12T^2 - 0.675T - 0.8 + 0.004c$$

where T is temperature in degrees Celsius above a constant reference temperature, which we refer to as the baseline. T is taken to be a function of time t in years, and c is the global average atmospheric CO_2 level in parts per million (ppm).

During the last 800,000 years, and before the industrial revolution, global average atmospheric CO_2 fluctuated between about 180 and 280 ppm.[1] Let us look first at the case where $c = 200$ ppm, which was a typical level during the ice ages.

1. Let us say that during a time when $c = 200$ ppm, there is a major global event that heats the earth by 0.25°C in a very short period of time (months to years), such as a strong El Niño, or a peak of the solar irradiance cycle. Use a numerical solver, such as Matlab's `ode45`, to solve for $T(t)$ after this event. (In other words, take $T(0) = 0.25$.) What temperature does the earth approach after some time after the event? How long does it take for the earth to cool down to within 0.1°C of its baseline?

2. Draw a phase plane with $\frac{dT}{dt}$ on the vertical axis and T on the horizontal axis. Identify the equilibrium points: the places where $\frac{dT}{dt} = 0$. Find the stability of the equilibrium points by considering values of T slightly above and below the equilibrium point. Argue why your answer to Question **1** makes sense in light of this.

3. Now consider another time when $c = 275$ ppm, a value seen historically during interglacial warm periods. Again draw the phase plane and identify the equilibrium points, along with their stability. Use a calculator to find numerical values for the equilibrium points. What is the meaning of these numbers?

4. Again, use a numerical solver to solve for $T(t)$ with $T(0) = 0.25$, but now use $c = 275$ ppm. How does this result differ from your answer to Question **1**? Does this make sense? Would a major global event that heats the earth by 0.25°C still mean that $T(0) = 0.25$? Find the answer again with a different initial condition that reflects such a situation, and confirm that the result matches your intuition.

5. In 2018, the global average atmospheric CO_2 was about $c = 407$ ppm. Draw the phase plane again, find all equilibrium points using a calculator, and determine the stability of each. How is this different from your answers to Questions **2** and **3**?

6. Imagine that a rare and catastrophic short-term warming event happened today and that it heated the earth by much more than 0.25°C; let's say it heated the earth by an amount b°C (above the lowest stable equilibrium point found in Question **5**). Is there a value of b beyond which the earth would not be able to recover back to temperatures within a few degrees of baseline? Find the value of b using a calculator. If such an event happened, what would happen to the earth's temperature? Where would it settle? Let us call this point "cataclysmic heat."

7. You should see a pattern in your phase plane graphs from Questions **2**, **3**, and **5**. If c were to keep increasing, is there a point beyond which the only stable equilibrium point is cataclysmic heat? Use a calculator to find this value of c. What would happen if c were to increase above this level? Some argue that this is what would happen if the ice at earth's poles melted completely.

8. Let's say that CO_2 levels rose briefly above c and the temperature reached cataclysmic heat. If the governments of the world, at this point, realized what was happening and developed a way to make CO_2 levels drop back below c again, would the earth be able to recover from the cataclysmic heat situation?

REFERENCES

[1] Earth System Research Laboratory-Global Monitoring Division. 2013. *CO_2 at NOAA's Mauna Loa observatory reaches new milestone: tops 400 ppm.* https://www.esrl.noaa.gov/gmd/news/7074.html.