Ejercicios 14.6
Integrales triples y aplicaciones

En los ejercicios 1 a 8, evaluar la integral iterada.

1) $\int_0^3\int_0^2\int_0^1 (x+y+z)\, dx\, dy\, dz$

$I = \int_0^1 x+y+z\, dx = \frac{x^2}{2} + yx + zx \Big|_0^1 = \frac{1}{2} + y + z$

$\int_0^2 \left(\frac{1}{2} + y + z\right) dy$

$= \frac{1}{2}y + \frac{y^2}{2} + zy \Big|_0^2 = 1 + 2 + 2z$

$= 2z + 3$

$\int_0^3 2z+3\, dz$

$= \frac{2z^2}{2} + 3z \Big|_0^3 = 3^2 + 3(3)$

$= 9 + 9$

$= \boxed{18}$

$=$

5) $\int_1^4 \int_0^1 \int_0^x 2ze^{-x^2}\, dy\, dx\, dz$

$I = \int_0^x 2ze^{-x^2}\, dy = 2ze^{-x^2} y \Big|_0^x = 2ze^{-x^2} x$

$\int_0^1 2ze^{-x^2} x\, dx = 2z \int_0^1 e^{-x^2} x\, dx$

$\left.\begin{array}{l} u = -x^2 \\ du = -2x\, dx \\ -\frac{du}{2} = x\, dx \end{array}\right\} -z \int_p^q e^u\, du = -z \left[e^u \right]_p^q$

$= -z \left[e^{-x^2} \right]_0^1 = -z \left[e^{-(1)^2} - e^{-(0)^2} \right]$

$= -z \left[\frac{1}{e} - 1 \right]$

$\int_1^4 -z \left[\frac{1}{e} - 1 \right] dz$

$\frac{1}{e} \int_1^4 -z\, dz + \int_1^4 z\, dz$

$\frac{1}{e} \left[\frac{-z^2}{2} \right]_1^4 + \frac{z^2}{2} \Big|_1^4$

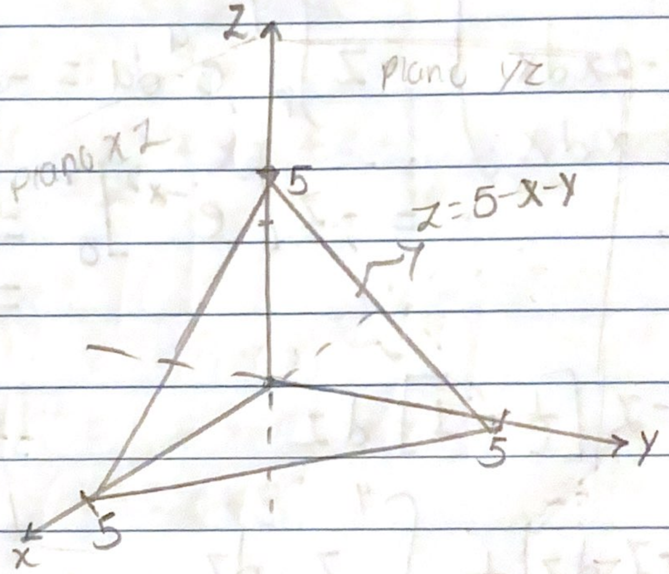
$= \frac{1}{e} \left[-8 + \frac{1}{2} \right] + \left[8 - \frac{1}{2} \right]$

$\frac{-15}{2e} + \left[\frac{15}{2} \right]$

$= \frac{15}{2} \left[1 - 1/e \right]$

En los ejercicios 13 a 18, dar una integral triple para el volumen del sólido.

13) El sólido en el primer octante acotado por los planos coordenados y el plano $z = 5 - x - y$



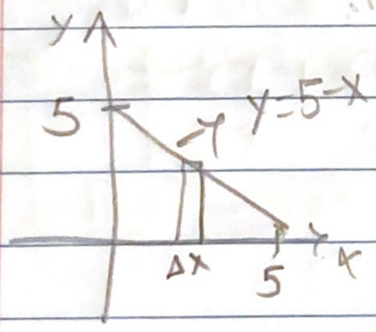
$$V = \iiint_E dV$$

$z = 5 - x - y$

Int - z = 5

Int - x = 5

int - y = 5



$0 \le z \le 5 - x - y$

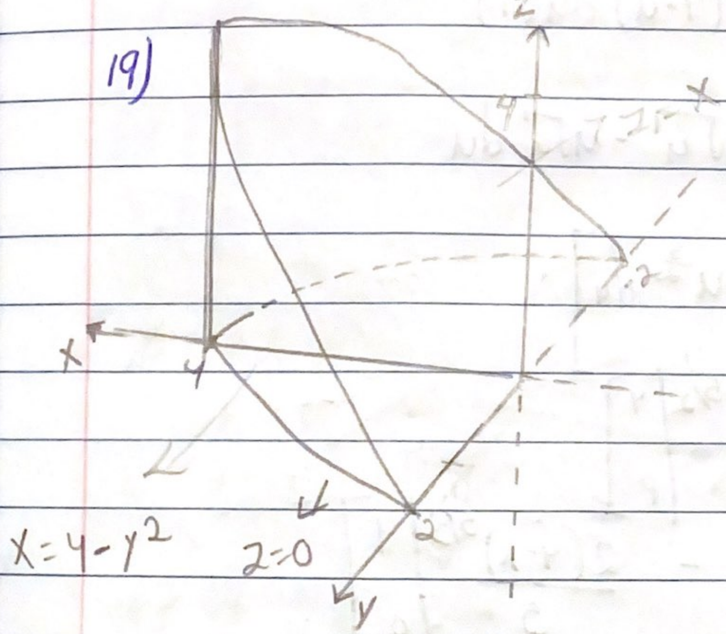
$0 \le y \le 5 - x$

$0 \le x \le 5$

$$V = \int_0^5 \int_0^{5-x} \int_0^{5-x-y} dz\,dy\,dx$$

Volumen. En los ejercicios 19 a 22, utilizar una integral triple para hallar el volumen del sólido mostrado en la figura.

19)



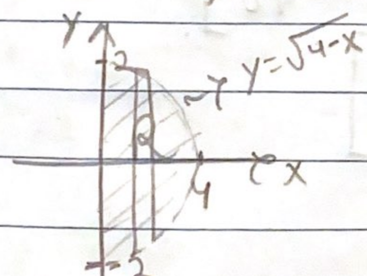
$V = \iiint_E dV$

$x = 4 - y^2$ $\quad z = 0$

R: $0 \le z \le x$

$-\sqrt{4-x} \le y \le \sqrt{4-x}$

$0 \le x \le 4$



$$\int_0^4 \int_{-\sqrt{4-x}}^{\sqrt{4-x}} \int_0^x dz\, dy\, dx$$

$$I = \int_0^x dz = z\Big|_0^x = x$$

$$\int_{-\sqrt{4-x}}^{\sqrt{4-x}} x\, dy = xy\Big|_{-\sqrt{4-x}}^{\sqrt{4-x}} = x(\sqrt{4-x}) - x(-\sqrt{4-x})$$

$$= x(\sqrt{4-x}) + x(\sqrt{4-x})$$

$$= 2x\sqrt{4-x}$$

$$\int_0^4 2x\sqrt{4-x}\, dx =$$

$$2\int_0^4 \sqrt{4-x}\, x\, dx =$$

$$2\int_0^4 x\sqrt{4-x}\,dx$$

$$\left.\begin{array}{l} u = 4-x \\ du = -dx \end{array}\right\} 2\int_\rho^9 \sqrt{u}\,(4-u)\,du\,(-1)$$

$$= -2\int_\rho^9 4\sqrt{u} - u\sqrt{u}\,du$$

$$= -2\left[4\int_\rho^9 u^{\frac{1}{2}}du - \int_\rho^9 u^{\frac{3}{2}}\,du\right]$$

$$= -2\left[4\left[\frac{2u^{3/2}}{3}\right]_\rho^9 - \frac{2u^{5/2}}{5}\Big|_\rho^9\right]$$

$$= -2\left[4\left[\frac{2(4-x)^{3/2}}{3}\right]_0^4 - \frac{2(4-x)^{5/2}}{5}\Big|_0^4\right]$$

$$= -2\left[4\left[0 - \frac{16}{3}\right] - \left(0 - \frac{64}{5}\right)\right]$$

$$= -2\left[-\frac{64}{3} + \frac{64}{5}\right]$$

$$\frac{128}{3} - \frac{128}{5} = \boxed{\frac{256}{15}}$$

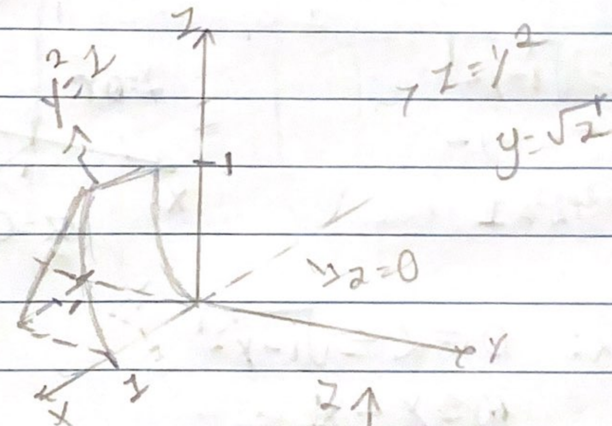
En los ejercicios 27 a 32, dibujar el sólido cuyo volumen está dado por la integral iterada y reescribir la integral utilizando el orden de integración indicado.

27) $\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\,dy\,dx$; Reescribir usando el orden $dy\,dz\,dx$.

$0 \le z \le y^2$

$-1 \le y \le 0$

$0 \le x \le 1$

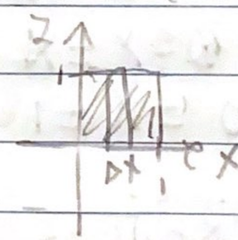


R:

$-1 \le y \le -\sqrt{z}$

$0 \le z \le 1$

$0 \le x \le 1$



$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\,dz\,dx$

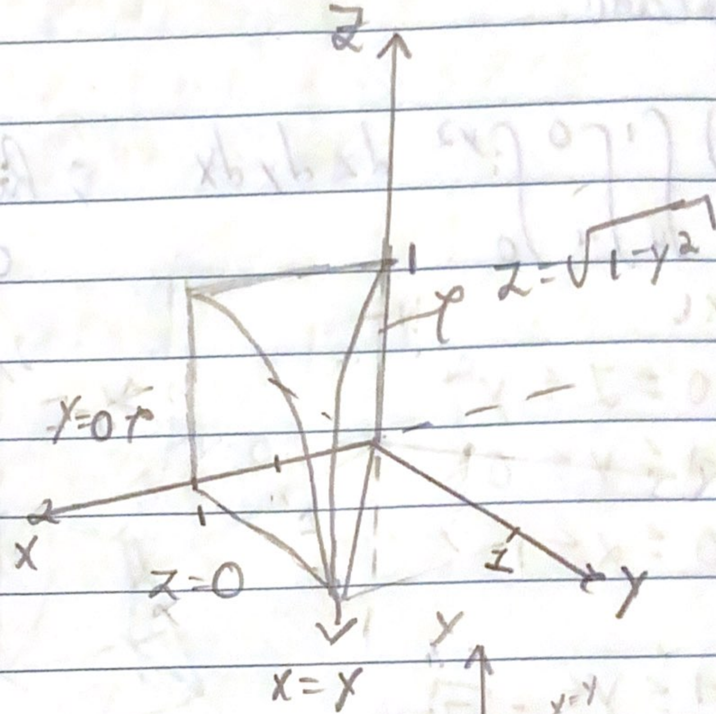
31) $\int_0^1 \int_y^1 \int_0^{\sqrt{1-y^2}} dz\,dx\,dy$

Reescribir utilizando el orden $dz\,dy\,dx$

$0 \leq z \leq \sqrt{1-y^2}$

$y \leq x \leq 1$

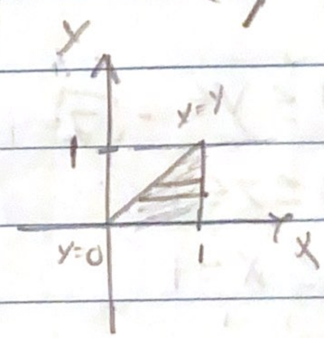
$0 \leq y \leq 1$



$z=\sqrt{1-y^2}$

$z^2 = 1-y^2$

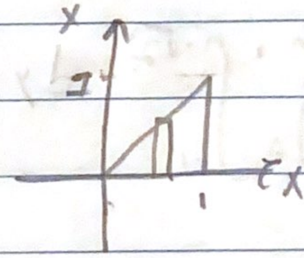
$y^2+z^2=1$



R: $0 \leq z \leq \sqrt{1-y^2}$

$0 \leq y \leq x$

$0 \leq x \leq 1$



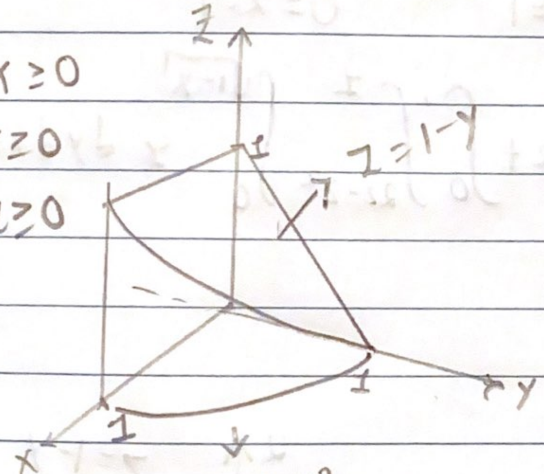
$$\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx$$

La región de integración de la integral dada
Reescribir la integral como una integral iterada
equivalente con los otros cinco órdenes.

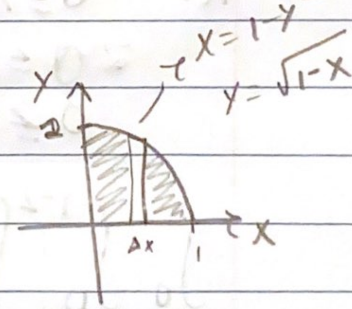
37) $\int_0^1 \int_0^{1-y^2} \int_0^{1-y} dz\,dx\,dy$

$R: 0 \le z \le 1-y$
$0 \le x \le 1-y^2$
$0 \le y \le 1$

$x \ge 0$
$y \ge 0$
$z \ge 0$

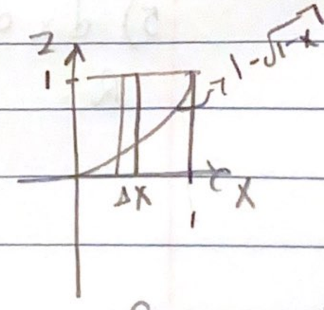


1) $R: 0 \le z \le 1-y$
$0 \le y \le \sqrt{1-x}$
$0 \le x \le 1$



$$\int_0^1 \int_0^{\sqrt{1-x}} \int_0^{1-y} dz\,dy\,dx$$

2) $R: 0 \le y \le 1-z$ $0 \le y \le \sqrt{1-x}$
$1-\sqrt{1-x} \le z \le 1$ + $0 \le z \le 1-\sqrt{1-x}$
$0 \le x \le 1$ $0 \le x \le 1$



$$\int_0^1 \int_{1-\sqrt{1-x}}^1 \int_0^{1-z} dy\,dz\,dx + \int_0^1 \int_0^{1-\sqrt{1-x}} \int_0^{\sqrt{1-x}} dy\,dz\,dx$$

inter $(z = 1-y,\ x = 1-y^2)$
$z = 1-\sqrt{1-x}$

inter$(z=1-y,\ x=1-y^2)$

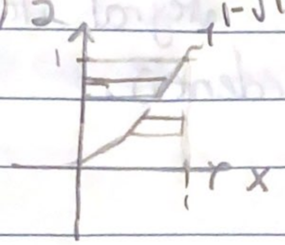
$x = 1-(z-1)^2$

$x = 1-(z^2-2z+1)$

$x = 2z-z^2$

dy dx dz

3)



R:

$0 \le y \le 1-z \qquad 0 \le y \le \sqrt{1-x}$

$0 \le x \le 1-(z-1)^2 \qquad 1-(z-1)^2 \le x \le 1$

$0 \le z \le 1 \qquad 0 \le z \le 1$

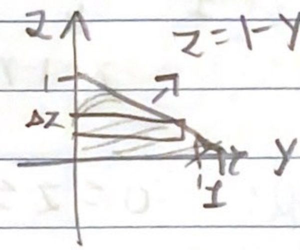
$$\int_0^1 \int_0^{2z-z^2} \int_0^{1-z} 1 \, dy\,dx\,dz + \int_0^1 \int_{2z-z^2}^{1} \int_0^{\sqrt{1-x}} 1 \, dy\,dx\,dz$$

4) dx dy dz

R: $0 \le x \le 1-y^2$

$0 \le y \le 1-z$

$0 \le z \le 1$



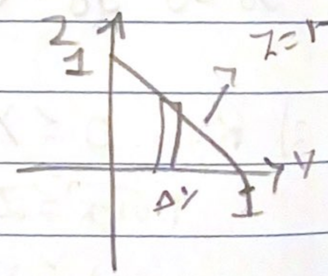
$$\int_0^1 \int_0^{1-z} \int_0^{1-y^2} dx\,dy\,dz$$

5) dx dz dy

R: $0 \le x \le 1-y^2$

$0 \le z \le 1-y$

$0 \le y \le 1$



$$\int_0^1 \int_0^{1-y} \int_0^{1-y^2} dx\,dz\,dy$$