4. Некоторые неопределенные интегралы

4.1. Интегралы от рациональных функций.

Интегралы, содержащие $X = ax + b$.

$$\int X^n\,dx = \frac{1}{a(n+1)} X^{n+1} \qquad (n \neq -1);$$

$$\int \frac{dx}{X} = \frac{1}{a}\ln|X|;$$

$$\int \frac{x\,dx}{X} = \frac{x}{a} - \frac{b}{a^2}\ln|X|;$$

$$\int \frac{x\,dx}{X^2} = \frac{b}{a^2X} + \frac{1}{a^2}\ln|X|;$$

$$\int \frac{x\,dx}{X^n} = \frac{1}{a^2}\left(\frac{-1}{(n-2)X^{n-2}} + \frac{b}{(n-1)X^{n-1}}\right) \qquad (n \neq 1, 2);$$

$$\int \frac{x^2dx}{X} = \frac{1}{a^3}\left(\frac{X^2}{2} - 2bX + b^2\ln|X|\right);$$

$$\int \frac{x^2dx}{X^2} = \frac{1}{a^3}\left(X - 2b\ln|X| - \frac{b^2}{X}\right);$$

$$\int \frac{x^2dx}{X^3} = \frac{1}{a^3}\left(\ln|X| + \frac{2b}{X} - \frac{b^2}{2X^2}\right);$$

$$\int \frac{x^2dx}{X^n} = \frac{1}{a^3}\left(\frac{-1}{(n-3)X^{n-3}} + \frac{2b}{(n-2)X^{n-2}} - \frac{b^2}{(n-1)X^{n-1}}\right) \qquad (n \neq 1, 2, 3);$$

$$\int \frac{dx}{xX} = -\frac{1}{b}\ln\left|\frac{X}{x}\right|;$$

$$\int \frac{dx}{xX^2} = -\frac{1}{b^2}\left(\ln\left|\frac{X}{x}\right| + \frac{ax}{X}\right);$$

$$\int \frac{dx}{xX^n} = -\frac{1}{b^n}\left[\ln\left|\frac{X}{x}\right| - \sum_{i=1}^{n-1} C_{n-1}^i \frac{(-a)^i x^i}{iX^i}\right] \qquad (n \geqslant 1);$$

$$\int \frac{dx}{x^2X} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{X}{x}\right|;$$

$$\int \frac{dx}{x^2X^2} = -a\left[\frac{1}{b^2X} + \frac{1}{ab^2x} - \frac{2}{b^3}\ln\left|\frac{X}{x}\right|\right];$$

$$\int \frac{dx}{x^2X^n} = -\frac{1}{b^{n+1}}\left[\sum_{i=2}^{n} C_n^i \frac{(-a)^{i-1}x^{i-1}}{(i-1)X^{i-1}} + \frac{X}{x} - na\ln\left|\frac{X}{x}\right|\right] \qquad (n \geqslant 2);$$

$$\int \frac{dx}{x^m X^n} = -\frac{1}{b^{m+n-1}} \sum_{i=0}^{m+n-2} C_{m+n-2}^{i} \frac{X^{m-i-1}(-a)^i}{(m-i-1)x^{m-i-1}};$$

если $m-i-1=0$, то соответствующий член под знаком суммы заменяется членом $C_{m+n-2}^{m-1}(-a)^{m-1}\ln\left|\frac{X}{x}\right|$.

Интегралы, содержащие $X = ax^2 + bx + c$ $(\Delta = 4ac - b^2)$.

$$\int \frac{dx}{X} = \begin{cases} \frac{2}{\sqrt{\Delta}} \operatorname{arctg} \frac{2ax+b}{\sqrt{\Delta}} & (\text{для } \Delta > 0), \\ -\frac{1}{\sqrt{-\Delta}} \ln\left|\frac{2ax+b-\sqrt{-\Delta}}{2ax+b+\sqrt{-\Delta}}\right| & (\text{для } \Delta < 0); \end{cases}$$

$$\int \frac{dx}{X^n} = \frac{2ax+b}{(n-1)\Delta X^{n-1}} + \frac{(2n-3)2a}{(n-1)\Delta}\int \frac{dx}{X^{n-1}};$$

$$\int \frac{x\,dx}{X} = \frac{1}{2a}\ln|X| - \frac{b}{2a}\int \frac{dx}{X};$$

$$\int \frac{x\,dx}{X^n} = -\frac{bx+2c}{(n-1)\Delta X^{n-1}} - \frac{b(2n-3)}{(n-1)\Delta}\int \frac{dx}{X^{n-1}};$$

$$\int \frac{x^2dx}{X} = \frac{x}{a} - \frac{b}{2a^2}\ln|X| + \frac{b^2-2ac}{2a^2}\int \frac{dx}{X};$$

$$\int \frac{x^2dx}{X^n} = \frac{-x}{(2n-3)\,aX^{n-1}} + \frac{c}{(2n-3)a}\int \frac{dx}{X^n} - \frac{(n-2)b}{(2n-3)a}\int \frac{x\,dx}{X^n};$$

$$\int \frac{x^m dx}{X^n} = -\frac{x^{m-1}}{(2n-m-1)\,aX^{n-1}} + \frac{(m-1)\,c}{(2n-m-1)\,a}\int \frac{x^{m-2}dx}{X^n} - \frac{(n-m)b}{(2n-m-1)\,a}\int \frac{x^{m-1}dx}{X^n} \quad (m \neq 2n-1);$$

$$\int \frac{x^{2n-1}dx}{X^n} = \frac{1}{a}\int \frac{x^{2n-3}dx}{X^{n-1}} - \frac{c}{a}\int \frac{x^{2n-3}dx}{X^n} - \frac{b}{a}\int \frac{x^{2n-2}dx}{X^n};$$

$$\int \frac{dx}{xX} = \frac{1}{2c}\ln\left|\frac{x^2}{X}\right| - \frac{b}{2c}\int \frac{dx}{X};$$

$$\int \frac{dx}{xX^n} = \frac{1}{2c(n-1)X^{n-1}} - \frac{b}{2c}\int \frac{dx}{X^n} + \frac{1}{c}\int \frac{dx}{xX^{n-1}};$$

$$\int \frac{dx}{x^2X} = \frac{b}{2c^2}\ln\left|\frac{X}{x^2}\right| - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right)\int \frac{dx}{X};$$

$$\int \frac{dx}{x^m X^n} = -\frac{1}{(m-1)\,cx^{m-1}X^{n-1}} - \frac{(2n+m-3)\,a}{(m-1)\,c}\int \frac{dx}{x^{m-2}X^n} - \frac{(n+m-2)\,b}{(m-1)\,c}\int \frac{dx}{x^{m-1}X^n} \quad (m > 1);$$

Интегралы, содержащие $X = a^2 \pm x^2$.

Через Y обозначено $\operatorname{arctg}\dfrac{x}{a}$ для знака плюс, $\operatorname{arth}\dfrac{x}{a} = \dfrac{1}{2}\ln\left|\dfrac{a+x}{a-x}\right|$ для знака минус при $|x| < a$, $\operatorname{arth}\dfrac{x}{a} = \dfrac{1}{2}\ln\left|\dfrac{a+x}{x-a}\right|$ для знака минус при $|x| > a$. В случае двойного знака в формуле верхний знак относится к $X = a^2 + x^2$, нижний знак — к $X = a^2 - x^2$.

$$\int \frac{dx}{X} = \frac{1}{a}Y;$$

$$\int \frac{dx}{X^{n+1}} \frac{x}{2na^2X^n} + \frac{2n-1}{2na^2}\int \frac{dx}{X^n};$$

$$\int \frac{x\,dx}{X} = \pm\frac{1}{2}\ln|X|;$$

$$\int \frac{x\,dx}{X^{n+1}} = \mp\frac{1}{2nX^n} \qquad (n \neq 0);$$

$$\int \frac{x^2dx}{X} = \pm x \mp aY;$$

$$\int \frac{x^2dx}{X^{n+1}} = \mp\frac{x}{2nX^n} \pm \frac{1}{2n}\int \frac{dx}{X^n} \qquad (n \neq 0);$$

$$\int \frac{x^3dx}{X} = \pm\frac{x^2}{2} - \frac{a^2}{2}\ln|X|;$$

$$\int \frac{x^3dx}{X^2} = \frac{a^2}{2X} + \frac{1}{2}\ln|X|;$$

$$\int \frac{x^3dx}{X^{n+1}} = -\frac{1}{2(n-1)X^{n-1}} + \frac{a^2}{2nX^n} \qquad (n > 1).$$

Интегралы, содержащие $X = a^3 \pm x^3$. В случае двойного знака в формуле верхний знак относится к $X = a^3 + x^3$, нижний знак — к $X = a^3 - x^3$.

$$\int \frac{dx}{X} = \pm\frac{1}{6a^2}\ln\frac{(a \pm x)^2}{\left|a^2 \mp ax + x^2\right|} + \frac{1}{a^2\sqrt{3}}\operatorname{arctg}\frac{2x \mp a}{a\sqrt{3}};$$

$$\int \frac{dx}{X^2} = \frac{x}{3a^3X} + \frac{3}{3a^3}\int \frac{dx}{X};$$

$$\int \frac{x\,dx}{X} = \frac{1}{6a}\ln\frac{\left|a^2 \mp ax + x^2\right|}{(a \pm x)^2} \pm \frac{1}{a\sqrt{3}}\operatorname{arctg}\frac{2x \mp a}{a\sqrt{3}};$$

$$\int \frac{x\,dx}{X^2} = \frac{x^2}{3a^2X} + \frac{1}{3a^2}\int \frac{x\,dx}{X};$$

$$\int \frac{x^2dx}{X} = \pm\frac{1}{3}\ln|X|;$$

$$\int \frac{x^2 dx}{X^2} = \mp \frac{1}{3X};$$

$$\int \frac{x^3 dx}{X} = \pm x \mp a^3 \int \frac{dx}{X};$$

$$\int \frac{x^3 dx}{X^2} = \mp \frac{x}{3X} \pm \frac{1}{3} \int \frac{dx}{X}.$$

Интегралы, содержащие $X = a^4 + x^4$.

$$\int \frac{dx}{a^4 + x^4} = \frac{1}{4a^3\sqrt{2}} \ln\left|\frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2}\right| + \frac{1}{2a^3\sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2};$$

$$\int \frac{x\,dx}{a^4 + x^4} = \frac{1}{2a^2} \operatorname{arctg} \frac{x^2}{a^2};$$

$$\int \frac{x^2 dx}{a^4 + x^4} = -\frac{1}{4a\sqrt{2}} \ln\left|\frac{x^2 + ax\sqrt{2} + a^2}{x^2 - ax\sqrt{2} + a^2}\right| + \frac{1}{2a\sqrt{2}} \operatorname{arctg} \frac{ax\sqrt{2}}{a^2 - x^2};$$

$$\int \frac{x^3 dx}{a^4 + x^4} = \frac{1}{4} \ln(a^4 + x^4).$$

Интегралы, содержащие $X = a^4 - x^4$.

$$\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3} \ln\left|\frac{a + x}{a - x}\right| + \frac{1}{2a^3} \operatorname{arctg} \frac{x}{a};$$

$$\int \frac{x\,dx}{a^4 - x^4} = \frac{1}{4a^3} \ln\left|\frac{a^2 + x^2}{a^2 - x^2}\right|;$$

$$\int \frac{x^2 dx}{a^4 - x^4} = \frac{1}{4a} \ln\left|\frac{a + x}{a - x}\right| - \frac{1}{2a} \operatorname{arctg} \frac{x}{a};$$

$$\int \frac{x^3 dx}{a^4 - x^4} = -\frac{1}{4} \ln\left|a^4 - x^4\right|.$$

4.2. Интегралы от иррациональных функций.

Интегралы вида $\int \frac{x^{\pm n+1/2} dx}{(a \pm bx)^m}$ (при $a > 0$, $b > 0$, $n = 0, 1, 2, \ldots$; $m = 1, 2, 3, \ldots$).

$$\int \frac{\sqrt{x}\,dx}{a + bx} = \frac{2\sqrt{x}}{b} - \frac{2\sqrt{a}}{b\sqrt{b}} \operatorname{arctg} \sqrt{\frac{bx}{a}};$$

$$\int \frac{\sqrt{x}\,dx}{a - bx} = -\frac{2\sqrt{x}}{b} + \frac{\sqrt{a}}{b\sqrt{b}} \ln\left|\frac{\sqrt{a} + \sqrt{bx}}{\sqrt{a} - \sqrt{bx}}\right|;$$

$$\int \frac{\sqrt{x}\,dx}{(a \pm bx)^m} = \frac{\pm x\sqrt{x}}{(m-1)\,a(a \pm bx)^{m-1}} + \frac{2m - 5}{2a(m-1)} \int \frac{\sqrt{x}\,dx}{(a \pm bx)^{m-1}} \qquad (m \geqslant 2);$$

$$\int \frac{x\sqrt{x}\,dx}{a+bx} = -\frac{6a\sqrt{x}-2bx\sqrt{x}}{3b^2} + \frac{2a^2}{b^2\sqrt{ab}}\operatorname{arctg}\sqrt{\frac{bx}{a}};$$

$$\int \frac{x\sqrt{x}\,dx}{a-bx} = -\frac{6a\sqrt{x}+2bx\sqrt{x}}{3b^2} + \frac{a\sqrt{a}}{b^2\sqrt{b}}\ln\left|\frac{\sqrt{a}+\sqrt{bx}}{\sqrt{a}-\sqrt{bx}}\right|;$$

$$\int \frac{x\sqrt{x}\,dx}{(a\pm bx)^m} = \pm\frac{x\sqrt{x}}{(m-1)(a\pm bx)^{m-1}} \mp \frac{3}{2b(m-1)}\int\frac{\sqrt{x}\,dx}{(a\pm bx)^{m-1}} \quad (m\geqslant 2);$$

$$\int \frac{x^n\sqrt{x}\,dx}{a\pm bx} = 2\sqrt{x}\sum_{k=0}^{n}\frac{(-1)^k a^k x^{n-k}}{(2n-2k+1)(\pm b)^{k+1}} + \frac{a^{n+1}}{(\mp b)^{n+1}}\int\frac{dx}{\sqrt{x}(a\pm bx)} \quad (n\geqslant 2);$$

$$\int \frac{dx}{\sqrt{x}(a+bx)} = \frac{2}{\sqrt{ab}}\operatorname{arctg}\sqrt{\frac{bx}{a}};$$

$$\int \frac{dx}{\sqrt{x}(a-bx)} = \frac{1}{\sqrt{ab}}\ln\left|\frac{\sqrt{a}+\sqrt{bx}}{\sqrt{a}-\sqrt{bx}}\right|;$$

$$\int \frac{dx}{\sqrt{x}\,(a+bx)^2} = \frac{\sqrt{x}}{a(a+bx)} + \frac{1}{a\sqrt{ab}}\operatorname{arctg}\sqrt{\frac{bx}{a}};$$

$$\int \frac{dx}{\sqrt{x}\,(a-bx)^2} = \frac{\sqrt{x}}{a(a-bx)} + \frac{1}{2a\sqrt{ab}}\ln\left|\frac{\sqrt{a}+\sqrt{bx}}{\sqrt{a}-\sqrt{bx}}\right|;$$

$$\int \frac{dx}{\sqrt{x}\,(a\pm bx)^m} = \frac{2\sqrt{x}}{a(a\pm bx)^{m-1}} \pm \frac{(2m-3)\,b}{a}\int\frac{\sqrt{x}\,dx}{(a\pm bx)^m}.$$

Интегралы вида $\int \frac{x^{\pm n}dx}{\sqrt{(a+bx)^m}}$ (при $n = 0, 1, 2, \ldots$; $m = 1, 3, 5, \ldots$).

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2}{b}\sqrt{a+bx};$$

$$\int \frac{x\,dx}{\sqrt{(a+bx)^m}} = \frac{2}{b^2\sqrt{(a+bx)^{m-2}}}\left(\frac{a+bx}{m-4} + \frac{a}{m-2}\right);$$

$$\int \frac{x^2\,dx}{\sqrt{(a+bx)^m}} = \frac{2}{b^3\sqrt{(a+bx)^{m-2}}}\left(-\frac{(a+bx)^2}{m-6} + \frac{2a(a+bx)}{m-4} - \frac{a^2}{m-2}\right);$$

$$\int \frac{x^3dx}{\sqrt{(a+bx)^m}} = \frac{2}{b^4\sqrt{(a+bx)^{m-2}}}\left[-\frac{(a+bx)^3}{m-8} + \frac{3a(a+bx)^2}{m-6} - \frac{3a^2(a+bx)}{m-4} + \frac{a^3}{m-2}\right];$$

$$\int \frac{x^n dx}{\sqrt{(a+bx)^m}} = \frac{2}{b^{n+1}\sqrt{(a+bx)^{m-2}}}\sum_{k=0}^{n}\frac{(-1)^k C_n^k (a+bx)^{n-k} a^k}{2n-2k-m+2};$$

$$\int \frac{dx}{x\sqrt{a+bx}} = \begin{cases} \dfrac{1}{\sqrt{a}} \ln\left|\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| & (a > 0), \\ \dfrac{2}{\sqrt{-a}} \operatorname{arctg} \dfrac{\sqrt{a+bx}}{\sqrt{-a}} & (a < 0); \end{cases}$$

$$\int \frac{dx}{x\sqrt{(a+bx)^m}} = \frac{2}{(m-2)\,a\sqrt{(a+bx)^{m-2}}} + \frac{1}{a}\int \frac{dx}{x\sqrt{(a+bx)^{m-2}}} \qquad (m \geqslant 3);$$

$$\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{(n-1)\,ax^{n-1}} - \frac{(2n-3)\,b}{2(n-1)\,a}\int \frac{dx}{x^{n-1}\sqrt{a+bx}} \qquad (n \geqslant 2);$$

$$\int \frac{dx}{x^n\sqrt{(a+bx)^m}} = -\frac{1}{(n-1)\,x^{n-1}\sqrt{(a+bx)^m}} - \frac{mb}{2(n-1)}\int \frac{dx}{x^{n-1}\sqrt{(a+bx)^{m+2}}} \qquad (n \geqslant 2);$$

$$= \frac{-2}{(m-2)\,bx^n\sqrt{(a+bx)^{m-2}}} - \frac{2n}{(m-2)\,b}\int \frac{dx}{x^{n+1}\sqrt{(a+bx)^{m-2}}} \qquad (m \geqslant 3).$$

Интегралы вида $\int x^{\pm n}\sqrt{(a+bx)^m}\,dx$ (при $n = 0, 1, 2, \ldots;$ $m = 1, 3, 5, \ldots$).

$$\int \sqrt{(a+bx)^m}\,dx = \frac{2\sqrt{(a+bx)^{m+2}}}{(m+2)\,b};$$

$$\int x\sqrt{(a+bx)^m}\,dx = \frac{2}{b^2}\left[\frac{\sqrt{(a+bx)^{m+4}}}{m+4} - \frac{a\sqrt{(a+bx)^{m+2}}}{m+2}\right];$$

$$\int x^2\sqrt{(a+bx)^m}\,dx = \frac{2}{b^3}\left[\frac{\sqrt{(a+bx)^{m+6}}}{m+6} - \frac{2a\sqrt{(a+bx)^{m+4}}}{m+4} + \frac{a^2\sqrt{(a+bx)^{m+2}}}{m+2}\right];$$

$$\int x^n\sqrt{(a+bx)^m}\,dx = \frac{2\sqrt{(a+bx)^{m+2}}}{b^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k C_n^k (a+bx)^{n-k} a^k}{2n-2k+m+2};$$

$$\int \frac{\sqrt{a+bx}}{x}\,dx = \begin{cases} 2\sqrt{a+bx} + \sqrt{a}\ln\left|\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| & (a > 0), \\ 2\sqrt{a+bx} + \dfrac{2a}{\sqrt{-a}}\operatorname{arctg}\dfrac{\sqrt{a+bx}}{\sqrt{-a}} & (a < 0); \end{cases}$$

$$\int \frac{\sqrt{(a+bx)^m}}{x}\,dx = \frac{2(a+bx)^{m/2}}{m} + a\int \frac{(a+bx)^{m/2-1}}{x}\,dx;$$

$$\int \frac{\sqrt{(a+bx)^m}}{x^2}\,dx = -\frac{\sqrt{(a+bx)^{m+2}}}{2ax^2} + \frac{mb}{2a}\int \frac{\sqrt{(a+bx)^m}}{x}\,dx;$$

$$\int \frac{\sqrt{a+bx}}{x^n}\,dx = -\frac{\sqrt{(a+bx)^3}}{(n-1)\,ax^{n-1}} + \frac{(5-2n)\,b}{2a(n-1)}\int \frac{\sqrt{a+bx}}{x^{n-1}}\,dx \qquad (n \geqslant 2);$$

$$\int \frac{\sqrt{(a+bx)^m}}{x^n}\,dx = -\frac{\sqrt{(a+bx)^{m+2}}}{(n-1)\,ax^{n-1}} + \frac{b(m-2n+4)}{2a(n-1)}\int \frac{\sqrt{(a+bx)^m}}{x^{n-1}}\,dx \qquad (n \geqslant 2)\,;$$

Интегралы вида $\int x^{\pm n}\sqrt{(a+bx)^{\pm n}(c+fx)^{\pm m}}\,dx$ ($\Delta = af - bc \neq 0$, $n = 1, 3, 5, \ldots$; $m = 1, 3, 5, \ldots$).

$$\int \frac{dx}{\sqrt{(a+bx)(c+fx)}} = \begin{cases} \dfrac{-1}{\sqrt{-bf}}\arcsin\dfrac{2bfx+af+bc}{\Delta} & (bf<0),\\ \dfrac{2}{\sqrt{bf}}\ln\left|\sqrt{bf(a+bx)}+b\sqrt{c+fx}\right| & (bf>0),\\ \dfrac{2}{\sqrt{-bf}}\operatorname{arctg}\sqrt{\dfrac{-f(a+bx)}{b(c+fx)}} & (b>0,\ f<0); \end{cases}$$

$$\int \frac{dx}{\sqrt{(a+bx)(c+fx)^m}} = -\frac{2}{\Delta(m-2)}\left[\sqrt{\frac{a+bx}{(c+fx)^{m-2}}} + \right.$$

$$\left. + \frac{b(m-3)}{2}\int \frac{dx}{\sqrt{(a+bx)(c+fx)^{m-2}}}\right] \qquad (m \geqslant 3);$$

$$\int \sqrt{\frac{a+bx}{c+fx}}\,dx = \begin{cases} \dfrac{\sqrt{(a+bx)(c+fx)}}{f} + \dfrac{\Delta}{2f\sqrt{-bf}}\arcsin\dfrac{2bfx+af+bc}{\Delta} & (bf<0),\\ \dfrac{\sqrt{(a+bx)(c+fx)}}{f} - \dfrac{\Delta}{f\sqrt{bf}}\ln\left|\sqrt{bf(a+bx)}+b\sqrt{c+fx}\right| & (bf>0); \end{cases}$$

$$\int \sqrt{\frac{(a+bx)^n}{c+fx}}\,dx = \frac{2\sqrt{(a+bx)^n(c+fx)}}{(n+1)f} - \frac{n\Delta}{(n+1)f}\int \sqrt{\frac{(a+bx)^{n-2}}{c+fx}}\,dx\,;$$

$$\int \sqrt{\frac{a+bx}{(c+fx)^m}}\,dx = -\frac{2}{f(m-2)}\sqrt{\frac{a+bx}{(c+fx)^{m-2}}} + \frac{b}{f(m-2)}\int \frac{dx}{\sqrt{(a+bx)(c+fx)^{m-2}}} \qquad (m \geqslant 3)\,;$$

$$\int \sqrt{(a+bx)^n(c+fx)}\,dx = \frac{2\sqrt{(a+bx)^{n+2}(c+fx)}}{b(n+3)} + \frac{\Delta}{b(n+3)}\int \sqrt{\frac{(a+bx)^n}{c+fx}}\,dx\,.$$

Интегралы вида $\int \frac{x^n\,dx}{\sqrt{(a^2+b^2x^2)^m}}$ ($a>0$, $b>0$, $n = 0, 1, 2, \ldots$; $m = 1, 3, 5, \ldots$).

$$\int \frac{dx}{\sqrt{a^2+b^2x^2}} = \frac{1}{b}\ln\left|bx+\sqrt{(a^2+b^2x^2)^m}\right|;$$

$$\int \frac{dx}{\sqrt{(a^2+b^2x^2)^m}} = \frac{1}{a^{m-1}}\sum_{k=0}^{(m-3)/2}\frac{(-1)^k C^k_{(m-3)/2}\,b^{2k}x^{2k+1}}{(2k+1)\sqrt{(a^2+b^2x^2)^{2k+1}}} \qquad (m \geqslant 3)\,;$$

$$\int \frac{x\,dx}{\sqrt{a^2+b^2x^2}} = \frac{1}{b^2}\sqrt{a^2+b^2x^2}\,;$$

$$\int \frac{x\,dx}{\sqrt{(a^2+b^2x^2)^m}} = -\frac{1}{(m-2)\,b^2\sqrt{(a^2+b^2x^2)^{m-2}}} \qquad (m \geqslant 3)\,;$$

$$\int \frac{x^2dx}{\sqrt{a^2+b^2x^2}} = \frac{x\sqrt{a^2+b^2x^2}}{2b^2} - \frac{a^2}{2b^3}\ln\left|bx+\sqrt{a^2+b^2x^2}\right|;$$

$$\int \frac{x^2dx}{\sqrt{(a^2+b^2x^2)^3}} = -\frac{x}{b^2\sqrt{a^2+b^2x^2}} + \frac{1}{b^3}\ln\left|bx+\sqrt{a^2+b^2x^2}\right|.$$

Интегралы вида $\int \frac{dx}{x^n\sqrt{(a^2+b^2x^2)^m}}$ $(a>0,\ b>0,\ n=1, 2, 3, \ldots;\ m=1, 3, 5, \ldots)$.

$$\int \frac{dx}{x\sqrt{a^2+b^2x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+b^2x^2}}{bx}\right|;$$

$$\int \frac{dx}{x\sqrt{(a^2+b^2x^2)^m}} = \sum_{k=1}^{(m-1)/2}\frac{1}{(m-2k)\,a^{2k}\sqrt{(a^2+b^2x^2)^{m-2k}}} - \frac{1}{a^m}\ln\left|\frac{a+\sqrt{a^2+b^2x^2}}{bx}\right| \qquad (m \geqslant 3);$$

$$\int \frac{dx}{x^2\sqrt{a^2+b^2x^2}} = -\frac{\sqrt{a^2+b^2x^2}}{a^2x};$$

$$\int \frac{dx}{x^2\sqrt{(a^2+b^2x^2)^m}} = \frac{1}{a^{m+1}}\sum_{k=0}^{(m-1)/2}\frac{(-1)^k C_{(m-1)/2}^k\, b^{2k}x^{2k-1}}{(2k-1)\sqrt{(a^2+b^2x^2)^{2k-1}}}\,;$$

$$\int \frac{dx}{x^3\sqrt{a^2+b^2x^2}} = -\frac{\sqrt{a^2+b^2x^2}}{2a^2x^2} + \frac{b^2}{2a^3}\ln\left|\frac{a+\sqrt{a^2+b^2x^2}}{bx}\right|;$$

$$\int \frac{dx}{x^3\sqrt{(a^2+b^2x^2)^m}} = -\frac{1}{(m-2)b^2x^4\sqrt{(a^2+b^2x^2)^{m-2}}} - \frac{4}{(m-2)\,b^2}\int\frac{dx}{x^5\sqrt{(a^2+b^2x^2)^{m-2}}} \qquad (m \geqslant 3);$$

Интегралы вида $\int x^{\pm n}\sqrt{(a^2+b^2x^2)^m}\,dx$ $(a>0,\ b>0,\ n=0,1,2,\ldots;\ m=1,3,5,\ldots)$.

$$\int \sqrt{a^2+b^2x^2}\,dx = \frac{x\sqrt{a^2+b^2x^2}}{2} + \frac{a^2}{2b}\ln\left|bx+\sqrt{a^2+b^2x^2}\right|;$$

$$\int \sqrt{(a^2+b^2x^2)^m}\,dx = \frac{x}{m+1}\sqrt{(a^2+b^2x^2)^m} + \frac{ma^2}{m+1}\int \sqrt{(a^2+b^2x^2)^{m-2}}\,dx\,;$$

$$\int x\sqrt{(a^2+b^2x^2)^m}\,dx = \frac{\sqrt{(a^2+b^2x^2)^{m+2}}}{(m+2)\,b^2}\,;$$

$$\int x^2\sqrt{(a^2+b^2x^2)^m}\,dx = \frac{x\sqrt{(a^2+b^2x^2)^{m+2}}}{(m+3)\,b^2} - \frac{a^2}{(m+3)\,b^2}\int \sqrt{(a^2+b^2x^2)^m}\,dx\,;$$

$$\int x^n\sqrt{(a^2+b^2x^2)^m}\,dx = \frac{x^{n-1}\sqrt{(a^2+b^2x^2)^{m+2}}}{(m+n+1)\,b^2} - \frac{a^2(n-1)}{b^2(m+n+1)}\int x^{n-2}\sqrt{(a^2+b^2x^2)^m}\,dx\,;$$

$$\int \frac{\sqrt{a^2+b^2x^2}}{x}\,dx = \sqrt{a^2+b^2x^2} - a\ln\left|\frac{a+\sqrt{a^2+b^2x^2}}{bx}\right|;$$

$$\int \frac{\sqrt{(a^2+b^2x^2)^m}}{x}\,dx = \frac{\sqrt{(a^2+b^2x^2)^m}}{m} + a^2\int \frac{\sqrt{(a^2+b^2x^2)^{m-2}}}{x}\,dx\,;$$

$$\int \frac{\sqrt{a^2+b^2x^2}}{x^2}\,dx = -\frac{\sqrt{a^2+b^2x^2}}{x} + b\ln\left|bx+\sqrt{a^2+b^2x^2}\right|;$$

$$\int \frac{\sqrt{(a^2+b^2x^2)^m}}{x^2}\,dx = -\frac{\sqrt{(a^2+b^2x^2)^m}}{x} + mb^2\int \sqrt{(a^2+b^2x^2)^{m-2}}\,dx\,.$$

Интегралы вида $\int x^{\pm n}\sqrt{(a^2-b^2x^2)^m}\,dx$ $(a>0,\ b>0,\ n=0,1,2,\ldots;\ m=1,3,5,\ldots)$.

$$\int \sqrt{a^2-b^2x^2}\,dx = \frac{x\sqrt{a^2-b^2x^2}}{2} + \frac{a^2}{2b}\arcsin\frac{bx}{a}\,;$$

$$\int \sqrt{(a^2-b^2x^2)^m}\,dx = \frac{x}{m+1}\sqrt{(a^2-b^2x^2)^m} + \frac{ma^2}{m+1}\int \sqrt{(a^2-b^2x^2)^{m-2}}\,dx \qquad (m\geqslant 3)\,;$$

$$\int x\sqrt{a^2-b^2x^2}\,dx = -\frac{\sqrt{(a^2-b^2x^2)^3}}{3b^2}\,;$$

$$\int x\sqrt{(a^2-b^2x^2)^m}\,dx = -\frac{\sqrt{(a^2-b^2x^2)^{m+3}}}{(m+2)\,b^2}\,;$$

$$\int x^2\sqrt{a^2-b^2x^2}\,dx = \frac{2b^2x^3-a^2x}{8b^2}\sqrt{a^2-b^2x^2} + \frac{a^4}{8b^3}\arcsin\frac{bx}{a}\,;$$

$$\int x^2\sqrt{(a^2-b^2x^2)^m}\,dx = -\frac{x\sqrt{(a^2-b^2x^2)^{m+2}}}{(m+3)\,b^2} + \frac{a^2}{(m+3)\,b^2}\int \sqrt{(a^2-b^2x^2)^m}\,dx\,;$$

$$\int \frac{\sqrt{a^2-b^2x^2}}{x}\,dx = \sqrt{a^2-b^2x^2} - a\ln\left|\frac{a+\sqrt{a^2-b^2x^2}}{bx}\right|;$$

$$\int \frac{\sqrt{(a^2-b^2x^2)^m}}{x}\,dx = \frac{1}{m}\sqrt{(a^2-b^2x^2)^m} + a^2\int \frac{\sqrt{(a^2-b^2x^2)^{m-2}}}{x}\,dx \qquad (m \geqslant 3);$$

$$\int \frac{\sqrt{a^2-b^2x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-b^2x^2}}{x} - b\arcsin\frac{bx}{a};$$

$$\int \frac{\sqrt{(a^2-b^2x^2)^m}}{x^2}\,dx = a^2\int \frac{\sqrt{(a^2-b^2x^2)^{m-2}}}{x^2}\,dx - b^2\int \sqrt{(a^2-b^2x^2)^{m-2}}\,dx.$$

Интегралы вида $\displaystyle\int \frac{x^n dx}{\sqrt{(a^2-b^2x^2)^m}}$ ($a > 0$, $b > 0$, $n = 0, 1, 2, \ldots$; $m = 1, 3, 5, \ldots$).

$$\int \frac{dx}{\sqrt{a^2-b^2x^2}} = \frac{1}{b}\arcsin\frac{bx}{a};$$

$$\int \frac{dx}{\sqrt{(a^2-b^2x^2)^m}} = \frac{1}{a^{m-1}}\sum_{k=0}^{(m-3)/2} \frac{C_{(m-3)/2}^{k} b^{2k} x^{2k+1}}{(2k+1)\sqrt{(a^2-b^2x^2)^{2k+1}}} \qquad (m \geqslant 3);$$

$$\int \frac{x\,dx}{\sqrt{a^2-b^2x^2}} = -\frac{\sqrt{a^2-b^2x^2}}{b^2};$$

$$\int \frac{x\,dx}{\sqrt{(a^2-b^2x^2)^m}} = \frac{1}{(m-2)\,b^2\sqrt{(a^2-b^2x^2)^{m-2}}};$$

$$\int \frac{x^2dx}{\sqrt{a^2-b^2x^2}} = -\frac{x\sqrt{a^2-b^2x^2}}{2b^2} + \frac{a^2}{2b^3}\arcsin\frac{bx}{a};$$

$$\int \frac{x^2dx}{\sqrt{(a^2-b^2x^2)^3}} = \frac{x}{b^2\sqrt{a^2-b^2x^2}} - \frac{1}{b^3}\arcsin\frac{bx}{a};$$

$$\int \frac{x^2dx}{\sqrt{(a^2-b^2x^2)^m}} = \frac{1}{a^{m-3}}\sum_{k=0}^{(m-5)/2} \frac{C_{(m-5)/2} b^{2k} x^{2k+3}}{(2k+3)\sqrt{(a^2-b^2x^2)^{2k+3}}} \qquad (m \geqslant 5);$$

$$\int \frac{x^n dx}{\sqrt{a^2-b^2x^2}} = -\frac{x^{n-1}\sqrt{a^2-b^2x^2}}{b^2} + \frac{n-1}{b^2}\int x^{n-2}\sqrt{a^2-b^2x^2}\,dx;$$

$$\int \frac{x^{2k+1}dx}{\sqrt{(a^2-b^2x^2)^m}} = \frac{1}{2}\int \frac{t^k dt}{\sqrt{(a^2-b^2t^2)^m}} \qquad (t = x^2);$$

$$\int \frac{x^n dx}{\sqrt{(a^2-b^2x^2)^m}} = \frac{x^{n-1}}{b^2(m-2)\sqrt{(a^2-b^2x^2)^{m-2}}} -$$

$$-\frac{n-1}{b^2(m-2)}\int \frac{x^{n-2}dx}{\sqrt{(a^2-b^2x^2)^{m-2}}} \qquad (m \geqslant 3);$$

Интегралы вида $\int \frac{dx}{x^n\sqrt{(a^2-b^2x^2)^m}}$ $(a>0,\ b>0,\ n=0,1,2,\ldots;\ m=1,3,5,\ldots)$.

$$\int \frac{dx}{x\sqrt{a^2-b^2x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-b^2x^2}}{bx}\right|;$$

$$\int \frac{dx}{x\sqrt{(a^2-b^2x^2)^m}} = \sum_{k=1}^{(m-1)/2} \frac{1}{(m-2k)\,a^{2k}\sqrt{(a^2-b^2x^2)^{m-2k}}} - \frac{1}{a^m}\ln\left|\frac{a+\sqrt{a^2-b^2x^2}}{bx}\right| \quad (m\geqslant 3);$$

$$\int \frac{dx}{x^2\sqrt{a^2-b^2x^2}} = -\frac{\sqrt{a^2-b^2x^2}}{a^2x};$$

$$\int \frac{dx}{x^2\sqrt{(a^2-b^2x^2)^m}} = \frac{1}{a^{m+1}}\sum_{k=0}^{(m-1)/2} \frac{C_{(m-1)/2}^k\, b^{2k}x^{2k-1}}{(2k-1)\sqrt{(a^2-b^2x^2)^{2k-1}}} \quad (m\geqslant 3).$$

Интегралы вида $\int x^{\pm n}\sqrt{(b^2x^2-a^2)^{\pm m}}\,dx$ $(a>0,\ b>0,\ n=0,1,2,\ldots;\ m=1,3,5,\ldots)$.

$$\int \frac{dx}{\sqrt{b^2x^2-a^2}} = \frac{1}{b}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|;$$

$$\int \frac{dx}{\sqrt{(b^2x^2-a^2)^m}} = \frac{(-1)^{(m-1)/2}}{a^{m-1}}\sum_{k=0}^{(m-3)/2} \frac{(-1)^k C_{(m-3)/2}^k\, b^{2k}x^{2k+1}}{(2k+1)\sqrt{(b^2x^2-a^2)^{2k+1}}} \quad (m\geqslant 3);$$

$$\int \frac{x\,dx}{\sqrt{(b^2x^2-a^2)^m}} = -\frac{1}{(m-2)\,b^2\sqrt{(b^2x^2-a^2)^{m-2}}};$$

$$\int \frac{x^2dx}{\sqrt{b^2x^2-a^2}} = \frac{x\sqrt{b^2x^2-a^2}}{2b} + \frac{a^2}{2b^3}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|;$$

$$\int \frac{x^2dx}{\sqrt{(b^2x^2-a^2)^3}} = -\frac{x}{b^2\sqrt{b^2x^2-a^2}} + \frac{1}{b^3}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|;$$

$$\int \frac{x^2dx}{\sqrt{(b^2x^2-a^2)^m}} = \frac{(-1)^{(m-3)/2}}{a^{m-3}}\sum_{k=0}^{(m-5)/2} \frac{(-1)^k C_{(m-5)/2}^k\, b^{2k}x^{2k+3}}{(2k+3)\sqrt{(b^2x^2-a^2)^{2k+3}}} \quad (m\geqslant 5);$$

$$\int \frac{x^ndx}{\sqrt{b^2x^2-a^2}} = \frac{x^{n-1}\sqrt{b^2x^2-a^2}}{b^2} - \frac{n-1}{b^2}\int x^{n-2}\sqrt{b^2x^2-a^2}\,dx;$$

$$\int \frac{x^ndx}{\sqrt{(b^2x^2-a^2)^m}} = -\frac{x^{n-1}}{b^2(m-2)\sqrt{(b^2x^2-a^2)^{m-2}}} + \frac{n-1}{b^2(m-2)}\int \frac{x^{n-2}dx}{\sqrt{(b^2x^2-a^2)^{m-2}}} \quad (m\geqslant 3);$$

$$\int \frac{dx}{x\sqrt{b^2x^2-a^2}} = \frac{1}{2}\arccos\left|\frac{a}{bx}\right|;$$

$$\int \frac{dx}{x^2\sqrt{b^2x^2-a^2}} = \frac{\sqrt{b^2x^2-a^2}}{a^2x};$$

$$\int \frac{dx}{x^n\sqrt{(b^2x^2-a^2)^m}} = -\frac{1}{(m-2)\,b^2x^{n+1}\sqrt{(b^2x^2-a^2)^{m-2}}} - \\ -\frac{n+1}{(m-2)\,b^2}\int \frac{dx}{x^{n+2}\sqrt{(b^2x^2-a^2)^{m-2}}} \qquad (m \geqslant 3);$$

$$\int \sqrt{b^2x^2-a^2}\,dx = \frac{x\sqrt{b^2x^2-a^2}}{2} - \frac{a^2}{2b}\ln\left|bx+\sqrt{b^2x^2-a^2}\right|;$$

$$\int \sqrt{(b^2x^2-a^2)^m}\,dx = \frac{x}{m+1}\sqrt{(b^2x^2-a^2)^m} - \frac{ma^2}{m+1}\int \sqrt{(b^2x^2-a^2)^{m-2}}\,dx;$$

$$\int x^n\sqrt{(b^2x^2-a^2)^m}\,dx = \frac{x^{n-1}\sqrt{(b^2x^2-a^2)^{m+2}}}{(m+n+1)\,b^2} + \frac{(n-1)\,a^2}{(m+n+1)\,b^2}\int x^{n-2}\sqrt{(b^2x^2-a^2)^m}\,dx;$$

$$\int \frac{\sqrt{b^2x^2-a^2}}{x}\,dx = \sqrt{b^2x^2-a^2} - a\arccos\left|\frac{a}{bx}\right|;$$

$$\int \frac{\sqrt{(b^2x^2-a^2)^m}}{x^n}\,dx = -a^2\int \frac{\sqrt{(b^2x^2-a^2)^{m-2}}}{x^n}\,dx + b^2\int \frac{\sqrt{(b^2x^2-a^2)^{m-2}}}{x^{n-2}}\,dx.$$

4.3. Интегралы от тригонометрических функций.

Интегралы, содержащие синус ($n \geqslant 0$ — целое, $a \geqslant 0$ — действительное).

$$\int \sin^n ax\,dx = -\frac{\sin^{n-1}ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2}ax\,dx \qquad (n>0);$$

$$\int x^n \sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx \qquad (n>0);$$

$$\int \frac{\sin ax}{x}\,dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - \frac{(ax)^7}{7\cdot 7!} + \ldots = \sum_{k=0}^{\infty}\frac{(-1)^k(ax)^{2k+1}}{(2k+1)\cdot(2k+1)!};$$

$$\int \frac{\sin ax}{x^n}\,dx = -\frac{1}{n-1}\frac{\sin ax}{x^{n-1}} + \frac{a}{n-1}\int \frac{\cos ax}{x^{n-1}}\,dx;$$

$$\int \frac{dx}{\sin^n ax} = -\frac{1}{a(n-1)}\frac{\cos ax}{\sin^{n-1}ax} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2}ax};$$

$$\int \frac{x\,dx}{\sin ax} = \frac{1}{a^2}\left(ax + \frac{(ax)^3}{3\cdot 3!} + \frac{7(ax)^5}{3\cdot 5\cdot 5!} + \frac{31(ax)^7}{3\cdot 7\cdot 7!} + \frac{127(ax)^9}{3\cdot 5\cdot 9!} + \ldots + \right.$$

$$\left. + \frac{2(2^{2n-1}-1)}{(2n+1)!} B_n (ax)^{2n+1} + \ldots \right) \quad (B_n \text{ – числа Бернулли});$$

$$\int \frac{x\,dx}{\sin^2 ax} = -\frac{x}{a}\operatorname{ctg} ax + \frac{1}{a^2}\ln\left|\sin ax\right|;$$

$$\int \frac{x\,dx}{\sin^n ax} = -\frac{x\cos ax}{(n-1)\,a\sin^{n-1} ax} - \frac{1}{(n-1)(n-2)\,a^2\sin^{n-2} ax} +$$

$$+ \frac{n-2}{n-1}\int \frac{x\,dx}{\sin^{n-2} ax} \qquad (n > 2);$$

$$\int \frac{dx}{1 \pm \sin ax} = \mp\frac{1}{a}\operatorname{tg}\left(\frac{\pi}{4} \mp \frac{ax}{2}\right);$$

$$\int \frac{x\,dx}{1+\sin ax} = -\frac{x}{a}\operatorname{tg}\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\cos\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right|;$$

$$\int \frac{x\,dx}{1-\sin ax} = \frac{x}{a}\operatorname{ctg}\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{2}{a^2}\ln\left|\sin\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right|;$$

$$\int \frac{\sin ax\,dx}{1 \pm \sin ax} = \pm x + \frac{1}{a}\operatorname{tg}\left(\frac{\pi}{4} \mp \frac{ax}{2}\right);$$

$$\int \frac{dx}{\sin ax\,(1 \pm \sin ax)} = \frac{1}{a}\operatorname{tg}\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + \frac{1}{a}\ln\left|\operatorname{tg}\frac{ax}{2}\right|;$$

$$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\operatorname{tg}\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\operatorname{tg}^3\left(\frac{\pi}{4} - \frac{ax}{2}\right);$$

$$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\operatorname{ctg}\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\operatorname{ctg}^3\left(\frac{\pi}{4} - \frac{ax}{2}\right);$$

$$\int \frac{\sin ax\,dx}{(1+\sin ax)^2} = -\frac{1}{2a}\operatorname{tg}\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\operatorname{tg}^3\left(\frac{\pi}{4} - \frac{ax}{2}\right);$$

$$\int \frac{\sin ax\,dx}{(1-\sin ax)^2} = -\frac{1}{2a}\operatorname{ctg}\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\operatorname{ctg}^3\left(\frac{\pi}{4} - \frac{ax}{2}\right);$$

$$\int \frac{dx}{1+\sin^2 ax} = \frac{1}{2\sqrt{2}\,a}\arcsin\left(\frac{3\sin^2 ax - 1}{\sin^2 ax + 1}\right) = \frac{1}{a\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg} ax);$$

$$\int \frac{dx}{1-\sin^2 ax} = \int \frac{dx}{\cos^2 ax} = \frac{1}{a}\operatorname{tg} ax;$$

$$\int \sin ax \sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad (|a| \ne |b|);$$

$$\int \frac{dx}{b + c\sin ax} = \begin{cases} \dfrac{2}{a\sqrt{b^2 - c^2}} \operatorname{arctg} \dfrac{b \operatorname{tg} \dfrac{ax}{2} + c}{\sqrt{b^2 - c^2}} & (b^2 > c^2), \\[2ex] \dfrac{2}{a\sqrt{c^2 - b^2}} \ln \left| \dfrac{b \operatorname{tg} \dfrac{ax}{2} + c - \sqrt{c^2 - b^2}}{b \operatorname{tg} \dfrac{ax}{2} + c + \sqrt{c^2 - b^2}} \right| & (b^2 < c^2); \end{cases}$$

$$\int \frac{\sin ax\, dx}{b + c\sin ax} = \frac{x}{c} - \frac{b}{c} \int \frac{dx}{b + c\sin ax};$$

$$\int \frac{dx}{\sin ax\,(b + c\sin ax)} = \frac{1}{ab} \ln \left| \operatorname{tg} \frac{ax}{2} \right| - \frac{c}{b} \int \frac{dx}{b + c\sin ax};$$

$$\int \frac{dx}{(b + c\sin ax)^2} = \frac{c\cos ax}{a(b^2 - c^2)(b + c\sin ax)} + \frac{b}{b^2 - c^2} \int \frac{dx}{b + c\sin ax};$$

$$\int \frac{\sin ax\, dx}{(b + c\sin ax)^2} = \frac{b\cos ax}{a(c^2 - b^2)(b + c\sin ax)} + \frac{c}{c^2 - b^2} \int \frac{dx}{b + c\sin ax};$$

$$\int \frac{dx}{b^2 + c^2 \sin^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}} \operatorname{arctg} \frac{\sqrt{b^2 + c^2} \operatorname{tg} ax}{b} \qquad (b > 0);$$

$$\int \frac{dx}{b^2 - c^2 \sin^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}} \operatorname{arctg} \frac{\sqrt{b^2 - c^2} \operatorname{tg} ax}{b} \qquad (b^2 > c^2,\ b > 0),$$

$$= \frac{1}{2ab\sqrt{c^2 - b^2}} \ln \left| \frac{\sqrt{c^2 - b^2} \operatorname{tg} ax + b}{\sqrt{c^2 - b^2} \operatorname{tg} ax - b} \right| \qquad (b^2 < c^2,\ b > 0);$$

Интегралы, содержащие косинус.

$$\int \cos^n ax\, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n} \int \cos^{n-2} ax\, dx;$$

$$\int x^n \cos ax\, dx = \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax\, dx;$$

$$\int \frac{\cos ax}{x} dx = \ln(ax) - \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} - \frac{(ax)^6}{6 \cdot 6!} + \ldots = \ln(ax) + \sum_{k=1}^{\infty} \frac{(-1)^k (ax)^{2k}}{2k \cdot (2k)!};$$

$$\int \frac{\cos ax}{x^n} dx = -\frac{\cos ax}{(n-1)\, x^{n-1}} - \frac{a}{n-1} \int \frac{\sin ax}{x^{n-1}} dx \qquad (n \neq 1);$$

$$\int \frac{dx}{\cos ax} = \int \sec ax\, dx = \frac{1}{a} \ln \left| \left(\frac{\pi}{4} + \frac{ax}{2} \right) \right| = \frac{1}{a} \ln \left| \sec ax + \operatorname{tg} ax \right|;$$

$$\int \frac{dx}{\cos^n ax} = \frac{1}{a(n-1)} \frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1} \int \frac{dx}{\cos^{n-2} ax} \qquad (n > 1);$$

$$\int \frac{x\,dx}{\cos ax} = \frac{1}{a^2}\left(\frac{(ax)^2}{2} + \frac{(ax)^4}{4\cdot 2!} + \frac{5(ax)^6}{6\cdot 4!} + \frac{61(ax)^8}{8\cdot 6!} + \frac{1385(ax)^{10}}{10\cdot 8!} + \right.$$

$$\left. + \ldots + \frac{E_n (ax)^{2n+2}}{(2n+2)\cdot(2n)!} + \ldots\right) \quad (E_n \text{ – числа Эйлера});$$

$$\int \frac{x\,dx}{\cos^2 ax} = \frac{x}{a}\operatorname{tg} ax + \frac{1}{a^2}\ln|\cos ax|;$$

$$\int \frac{x\,dx}{\cos^n ax} = \frac{x\sin ax}{(n-1)a\cos^{n-1} ax} - \frac{1}{(n-1)(n-2)a^2\cos^{n-2} ax} +$$

$$+ \frac{n-2}{n-1}\int \frac{x\,dx}{\cos^{n-2} ax} \quad (n > 2);$$

$$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\operatorname{tg}\frac{ax}{2};$$

$$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\operatorname{ctg}\frac{ax}{2};$$

$$\int \frac{x\,dx}{1+\cos ax} = \frac{x}{a}\operatorname{tg}\frac{ax}{2} + \frac{2}{a^2}\ln\left|\cos\frac{ax}{2}\right|;$$

$$\int \frac{x\,dx}{1-\cos ax} = -\frac{x}{a}\operatorname{ctg}\frac{ax}{2} + \frac{2}{a^2}\ln\left|\sin\frac{ax}{2}\right|;$$

$$\int \frac{\cos ax\,dx}{1+\cos ax} = x - \frac{1}{a}\operatorname{tg}\frac{ax}{2};$$

$$\int \frac{\cos ax\,dx}{1-\cos ax} = -x - \frac{1}{a}\operatorname{ctg}\frac{ax}{2};$$

$$\int \frac{dx}{\cos ax(1+\cos ax)} = \frac{1}{a}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| - \frac{1}{a}\operatorname{tg}\frac{ax}{2};$$

$$\int \frac{dx}{\cos ax(1-\cos ax)} = \frac{1}{a}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right| - \frac{1}{a}\operatorname{ctg}\frac{ax}{2};$$

$$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\operatorname{tg}\frac{ax}{2} + \frac{1}{6a}\operatorname{tg}^3\frac{ax}{2};$$

$$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\operatorname{ctg}\frac{ax}{2} - \frac{1}{6a}\operatorname{ctg}^3\frac{ax}{2};$$

$$\int \frac{\cos ax\,dx}{(1+\cos ax)^2} = \frac{1}{2a}\operatorname{tg}\frac{ax}{2} - \frac{1}{6a}\operatorname{tg}^3\frac{ax}{2};$$

$$\int \frac{\cos ax\,dx}{(1-\cos ax)^2} = \frac{1}{2a}\operatorname{ctg}\frac{ax}{2} - \frac{1}{6a}\operatorname{ctg}^3\frac{ax}{2};$$

$$\int \frac{dx}{1+\cos^2 ax} = \frac{1}{2\sqrt{2}\,a}\arcsin\left(\frac{1-3\cos^2 ax}{1+\cos^2 ax}\right);$$

$$\int \frac{dx}{1-\cos^2 ax} = \int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\operatorname{ctg} ax;$$

$$\int \cos ax \cos bx\, dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad (|a| \neq |b|);$$

$$\int \frac{dx}{b + c\cos ax} = \frac{2}{\sqrt{b^2 - c^2}} \operatorname{arctg} \frac{(b-c)\operatorname{tg}\frac{ax}{2}}{\sqrt{b^2 - c^2}} \qquad (b^2 > c^2),$$

$$= \frac{1}{\sqrt{c^2 - b^2}} \ln \frac{(c-b)\operatorname{tg}\frac{ax}{2} + \sqrt{c^2 - b^2}}{(c-b)\operatorname{tg}\frac{ax}{2} - \sqrt{c^2 - b^2}} \qquad (b^2 < c^2);$$

$$\int \frac{\cos ax\, dx}{b + c\cos ax} = \frac{x}{c} - \frac{b}{c}\int \frac{dx}{b + c\cos ax};$$

$$\int \frac{dx}{\cos ax\,(b + c\cos ax)} = \frac{1}{ab}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{ax}{2}\right)\right| - \frac{c}{b}\int \frac{dx}{b + c\cos ax};$$

$$\int \frac{dx}{(b + c\cos ax)^2} = \frac{c\sin ax}{a(c^2 - b^2)(b + c\cos ax)} - \frac{b}{c^2 - b^2}\int \frac{dx}{b + c\cos ax};$$

$$\int \frac{\cos ax\, dx}{(b + c\cos ax)^2} = \frac{b\sin ax}{a(b^2 - c^2)(b + c\cos ax)} - \frac{c}{b^2 - c^2}\int \frac{dx}{b + c\cos ax};$$

$$\int \frac{dx}{b^2 + c^2\cos^2 ax} = \frac{1}{ab\sqrt{b^2 + c^2}} \operatorname{arctg} \frac{b\operatorname{tg} ax}{\sqrt{b^2 + c^2}} \qquad (b > 0);$$

$$\int \frac{dx}{b^2 - c^2\cos^2 ax} = \frac{1}{ab\sqrt{b^2 - c^2}} \operatorname{arctg} \frac{b\operatorname{tg} ax}{\sqrt{b^2 - c^2}} \qquad (b^2 > c^2,\ b > 0),$$

$$= \frac{1}{ab\sqrt{c^2 - b^2}} \ln\left|\frac{b\operatorname{tg} ax - \sqrt{c^2 - b^2}}{b\operatorname{tg} ax + \sqrt{c^2 - b^2}}\right| \qquad (b^2 < c^2,\ b > 0);$$

Интегралы, содержащие синус и косинус.

$$\int \frac{\sin x\, dx}{a + b\cos x} = -\frac{1}{b}\ln|a + b\cos x|;$$

$$\int \frac{\sin x\, dx}{(a + b\cos x)^n} = \frac{1}{(n-1)\,b\,(a + b\cos x)^{n-1}} \qquad (n \geqslant 2);$$

$$\int \frac{\sin x\, dx}{\cos x\,(1 \pm \cos x)} = \ln\left|\frac{1 \pm \cos x}{\cos x}\right|;$$

$$\int \frac{\sin x\, dx}{\cos x\,(1 \pm \sin x)} = \frac{1}{2(1 \pm \sin x)} \pm \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|;$$

$$\int \frac{\sin x\, dx}{(a + b\cos x)(c + d\cos x)} = \frac{1}{ad - bc}\ln\left|\frac{a + b\cos x}{c + d\cos x}\right| \qquad (ad - bc \neq 0);$$

$$\int \frac{c + d\sin x}{a + b\cos x}\,dx = -\frac{d}{b}\ln|a + b\cos x| + c\int \frac{dx}{a + b\cos x};$$

$$\int \frac{\cos x\, dx}{a + b\sin x} = \frac{1}{b}\ln|a + b\sin x|;$$

$$\int \frac{\cos x\, dx}{(a+b\sin x)^n} = -\frac{1}{(n-1)\,b\,(a+b\sin x)^{n-1}} \qquad (n \geqslant 2);$$

$$\int \frac{\cos x\, dx}{\sin x\,(1\pm\sin x)} = \ln\left|\frac{\sin x}{1\pm\sin x}\right|;$$

$$\int \frac{\cos x\, dx}{\sin x\,(1\pm\cos x)} = -\frac{1}{2(1\pm\cos x)} \pm \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|;$$

$$\int \frac{\cos x\, dx}{(a+b\sin x)(c+d\sin x)} = \frac{1}{ad-bc}\ln\left|\frac{c+d\sin x}{a+b\sin x}\right| \qquad (ad-bc\neq 0);$$

$$\int \frac{c+d\cos x}{a+b\sin x}\,dx = \frac{d}{b}\ln\left|a+b\sin x\right| + c\int\frac{dx}{a+b\sin x};$$

$$\int \frac{dx}{\sin x\,(1\pm\cos x)} = \pm\frac{1}{2\,(1\pm\cos x)} + \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|;$$

$$\int \frac{dx}{\cos x\,(1\pm\sin x)} = \mp\frac{1}{2\,(1\pm\cos x)} + \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|;$$

$$\int \frac{dx}{\sin x\pm\cos x} = \frac{1}{\sqrt{2}}\ln\left|\operatorname{tg}\left(\frac{x}{2}\pm\frac{\pi}{8}\right)\right|;$$

$$\int \frac{dx}{a\cos x+b\sin x} = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\frac{x+\arccos\frac{b}{\sqrt{a^2+b^2}}}{2}\right|;$$

$$\int \frac{dx}{(a\cos x+b\sin x)^n} = \int\frac{d(x-\varphi)}{\left[\rho\cos\left(x-\varphi\right)\right]^n} \qquad (a=\rho\cos\varphi,\ b=\rho\sin\varphi);$$

$$\int \frac{dx}{1+\cos x\pm\sin x} = \pm\ln\left|1\pm\operatorname{tg}\frac{x}{2}\right|;$$

$$\int \frac{dx}{a+b\cos x+c\sin x} = \int\frac{dt}{a+\sqrt{b^2+c^2}\,\sin t} \qquad \left(t = x+\operatorname{arctg}\frac{b}{c}\right);$$

$$\int \frac{dx}{(a+b\cos x+c\sin x)^n} = \int\frac{d(x-\varphi)}{\left[a+\rho\cos\left(x-\varphi\right)\right]^n} \qquad (b=\rho\cos\varphi,\ c=\rho\sin\varphi);$$

$$\int \frac{\sin x\, dx}{\sin x\pm\cos x} = \frac{x}{2}\mp\frac{1}{2}\ln\left|\sin x\pm\cos x\right|;$$

$$\int \frac{\cos x\, dx}{\sin x\pm\cos x} = \pm\frac{x}{2}+\frac{1}{2}\ln\left|\sin x\pm\cos x\right|;$$

$$\int \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} = \frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}x\right) \qquad (a>0,\ b>0);$$

$$\int \frac{dx}{a^2\cos^2 x-b^2\sin^2 x} = \frac{1}{2ab}\ln\left|\frac{b\operatorname{tg}x+a}{b\operatorname{tg}x-a}\right|;$$

$$\int \frac{\sin x\cos x\, dx}{a\cos^2 x+b\sin^2 x} = \frac{1}{2\left(b-a\right)}\ln\left|a\cos^2 x+b\sin^2 x\right| \qquad (a\neq b).$$

Интегралы, содержащие тангенс и котангенс.

$$\int \operatorname{tg} x\, dx = -\ln|\cos x|;$$

$$\int \operatorname{tg}^n x\, dx = \frac{\operatorname{tg}^{n-1} x}{n-1} - \int \operatorname{tg}^{n-2} x\, dx \qquad (n \geqslant 2);$$

$$\int \operatorname{ctg} x\, dx = \ln|\sin x|;$$

$$\int \operatorname{ctg}^n x\, dx = -\frac{\operatorname{ctg}^{n-1} x}{n-1} - \int \operatorname{ctg}^{n-2} x\, dx \qquad (n \geqslant 2);$$

$$\int \frac{dx}{\operatorname{tg} x \pm 1} = \pm\frac{x}{2} + \frac{1}{2}\ln|\sin x \pm \cos x|;$$

$$\int \frac{dx}{a + b\operatorname{tg} x} = \frac{1}{a^2 + b^2}\left(b\ln|a + b\operatorname{tg} x| + b\ln|\cos x| + ax\right);$$

$$\int \frac{\operatorname{tg} x\, dx}{\operatorname{tg} x \pm 1} = \frac{x}{2} \mp \frac{1}{2}\ln|\sin x \pm \cos x|;$$

$$\int \frac{\operatorname{tg} x\, dx}{a + b\operatorname{tg} x} = \frac{1}{a^2 + b^2}\left(bx - a\ln|a\cos x + b\sin x|\right);$$

$$\int \frac{dx}{1 + \operatorname{tg}^2 x} = \frac{x}{2} + \frac{1}{4}\sin 2x;$$

$$\int \frac{dx}{a^2 + b^2\operatorname{tg}^2 x} = \frac{1}{a^2 - b^2}\left[x - \left|\frac{b}{a}\right|\operatorname{arctg}\left(\left|\frac{b}{a}\right|\operatorname{tg} x\right)\right] \qquad \left(a^2 \neq b^2\right);$$

$$\int \frac{dx}{a^2 - b^2\operatorname{tg}^2 x} = \frac{1}{a^2 + b^2}\left(x + \frac{b}{2a}\ln\left|\frac{a + b\operatorname{tg} x}{a - b\operatorname{tg} x}\right|\right);$$

$$\int \frac{\operatorname{tg} x\, dx}{1 + \operatorname{tg}^2 x} = -\frac{\cos^2 x}{2};$$

$$\int \frac{\operatorname{tg} x\, dx}{1 + a^2\operatorname{tg}^2 x} = \frac{\ln\left(\cos^2 x + a\sin^2 x\right)}{2\left(a^2 - 1\right)} \qquad \left(a^2 \neq 1\right);$$

$$\int \frac{dx}{\operatorname{ctg} x \pm 1} = \frac{x}{2} \pm \frac{1}{2}\ln|\sin x \pm \cos x|;$$

$$\int \frac{\operatorname{ctg} x\, dx}{\operatorname{ctg} x \pm 1} = \pm\frac{x}{2} + \frac{1}{2}\ln|\sin x \pm \cos x|;$$

$$\int \frac{\operatorname{ctg} x\, dx}{a + b\operatorname{ctg} x} = \int \frac{dx}{a\operatorname{tg} x + b};$$

$$\int \frac{dx}{1 + \operatorname{ctg}^2 x} = \frac{x}{2} - \frac{1}{4}\sin 2x;$$

$$\int \frac{dx}{a^2 + b^2\operatorname{ctg}^2 x} = \frac{1}{a^2 - b^2}\left[x - \left|\frac{b}{a}\right|\operatorname{arctg}\left(-\left|\frac{b}{a}\right|\operatorname{ctg} x\right)\right] \qquad \left(a^2 \neq b^2\right);$$

$$\int \frac{dx}{a^2 - b^2\operatorname{ctg}^2 x} = \frac{1}{a^2 + b^2}\left(x + \frac{b}{2a}\ln\left|\frac{a - b\operatorname{ctg} x}{a + b\operatorname{ctg} x}\right|\right);$$

$$\int \frac{\operatorname{ctg} x\,dx}{1+\operatorname{ctg}^2 x} = \frac{\sin^2 x}{2};$$

$$\int \frac{\operatorname{ctg} x\,dx}{1+a^2\operatorname{ctg}^2 x} = \frac{1}{a^2}\int \frac{\operatorname{tg} x\,dx}{1+\frac{1}{a^2}\operatorname{tg}^2 x};$$

$$\int \frac{\operatorname{tg} x\,dx}{1\pm\operatorname{ctg} x} = \mp\frac{x}{2}+\frac{1}{2}\ln\left|\frac{\sin x\pm\cos x}{\cos^2 x}\right|;$$

$$\int \frac{\operatorname{tg} x\,dx}{\sqrt{a+b\operatorname{tg}^2 x}} = \frac{1}{\sqrt{b-a}}\arccos\left(\frac{\sqrt{b-a}}{\sqrt{b}}\cos x\right);$$

$$\int x\operatorname{tg} ax\,dx = \frac{ax^3}{3}+\frac{a^3x^5}{15}+\frac{2a^5x^7}{105}+\frac{17a^7x^9}{2835}+\ldots+\frac{2^{2n}\left(2^{2n}-1\right)B_n\,a^{2n-1}x^{2n+1}}{(2n+1)!},$$

$(B_n$ – числа Бернулли$)$;

$$\int \frac{\operatorname{tg} ax\,dx}{x} = ax+\frac{(ax)^3}{9}+\frac{2(ax)^5}{75}+\frac{17(ax)^7}{2205}+\ldots+\frac{2^{2n}\left(2^{2n}-1\right)B_n\,(ax)^{2n-1}}{(2n-1)(2n)!},$$

$(B_n$ – числа Бернулли$)$;

$$\int x\operatorname{ctg} ax\,dx = \frac{x}{a}-\frac{ax^3}{9}-\frac{a^3x^5}{225}-\ldots-\frac{2^{2n}B_n\,a^{2n-1}x^{2n+1}}{(2n+1)!},\quad (B_n \text{ – числа Бернулли});$$

$$\int \frac{\operatorname{ctg} ax\,dx}{x} = -\frac{1}{ax}-\frac{ax}{3}-\frac{(ax)^3}{135}-\frac{2(ax)^5}{4725}-\ldots-\frac{2^{2n}B_n\,(ax)^{2n-1}}{(2n-1)(2n)!},$$

$(B_n$ – числа Бернулли$)$.

4.4. Интегралы, содержащие показательную функцию.

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax};$$

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax}-\frac{x}{a}\int x^{n-1}e^{ax}\,dx;$$

$$\int \frac{e^{ax}}{x}dx = \ln x+\frac{ax}{1\cdot 1!}+\frac{(ax)^2}{2\cdot 2!}+\frac{(ax)^3}{3\cdot 3!}+\ldots = \ln x+\sum_{k=1}^{\infty}\frac{(ax)^k}{k\cdot k!};$$

$$\int \frac{e^{ax}}{x^n}dx = \frac{1}{n-1}\left(-\frac{e^{ax}}{x^{n-1}}+a\int\frac{e^{ax}}{x^{n-1}}dx\right)\quad (n\neq 1);$$

$$\int \frac{dx}{1+e^{ax}} = \frac{1}{a}\ln\frac{e^{ax}}{1+e^{ax}};$$

$$\int \frac{dx}{b+ce^{ax}} = \frac{x}{b}-\frac{1}{ab}\ln\left(b+ce^{ax}\right);$$

$$\int \frac{e^{ax}dx}{b+ce^{ax}} = \frac{1}{ac}\ln\left(b+ce^{ax}\right);$$

$$\int e^{ax}\ln x\,dx = \frac{e^{ax}\ln x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}dx;$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right);$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right);$$

$$\int e^{ax}\sin^n x\,dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}\left(a\sin x - n\cos x\right) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2}x\,dx;$$

$$\int e^{ax}\cos^n x\,dx = \frac{e^{ax}\cos^{n-1}x}{a^2+n^2}\left(a\cos x + n\sin x\right) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2}x\,dx;$$

$$\int x e^{ax}\sin bx\,dx = \frac{x e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) - \\ - \frac{e^{ax}}{(a^2+b^2)^2}\left[(a^2-b^2)\sin bx - 2ab\cos bx\right];$$

$$\int x e^{ax}\cos bx\,dx = \frac{x e^{ax}}{a^2+b^2}\left(a\cos bx - b\sin bx\right) - \\ - \frac{e^{ax}}{(a^2+b^2)^2}\left[(a^2-b^2)\cos bx - 2ab\sin bx\right].$$

4.5. Интегралы, содержащие логарифмическую функцию.

$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1}dx \qquad (n \neq -1);$$

$$\int \frac{dx}{\ln x} = \ln|\ln x| + \ln x + \frac{(\ln x)^2}{2\cdot 2!} + \frac{(\ln x)^3}{3\cdot 3!} + \ldots = \ln|\ln x| + \sum_{k=1}^{\infty}\frac{(\ln x)^k}{k\cdot k!};$$

$$\int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1)(\ln x)^{n-1}} + \frac{1}{n-1}\int \frac{dx}{(\ln x)^{n-1}} \qquad (n \neq 1);$$

$$\int x^m \ln x\,dx = x^{m+1}\left[\frac{\ln x}{m+1} - \frac{1}{(m+1)^2}\right] \qquad (m \neq -1);$$

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1}dx \qquad (m \neq 1,\ n \neq -1);$$

$$\int \frac{(\ln x)^n}{x}dx = \frac{(\ln x)^{n-1}}{n+1};$$

$$\int \frac{\ln x}{x^m}dx = -\frac{\ln x}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2 x^{m-1}} \qquad (m \neq 1);$$

$$\int \frac{(\ln x)^n}{x^m}dx = -\frac{(\ln x)^n}{(m-1)x^{m-1}} + \frac{n}{m-1}\int \frac{(\ln x)^{n-1}}{x^m}dx \qquad (m \neq 1);$$

$$\int \frac{x^m\,dx}{(\ln x)^n} = -\frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\ln x)^{n-1}} \quad (n \neq 1);$$

$$\int \frac{dx}{x\ln x} = \ln|\ln x|;$$

$$\int \frac{dx}{x^n \ln x} = \ln|\ln x| - (n-1)\ln x + \frac{(n-1)^2(\ln x)^2}{2\cdot 2!} - \frac{(n-1)^3(\ln x)^3}{3\cdot 3!} + \ldots =$$

$$= \ln|\ln x| + \sum_{k=1}^{\infty} \frac{\left[(-1)(n-1)\ln x\right]^k}{k\cdot k!};$$

$$\int \frac{dx}{x(\ln x)^n} = -\frac{1}{(n-1)(\ln x)^{n-1}} \quad (n \neq 1);$$

$$\int \frac{dx}{x^m(\ln x)^n} = -\frac{1}{x^{m-1}(n-1)(\ln x)^{n-1}} - \frac{m-1}{n-1}\int \frac{dx}{x^m(\ln x)^{n-1}} \quad (n \neq 1);$$

$$\int \sin\ln x\,dx = \frac{x}{2}(\sin\ln x - \cos\ln x);$$

$$\int \cos\ln x\,dx = \frac{x}{2}(\sin\ln x + \cos\ln x);$$

$$\int e^{ax}\ln x\,dx = \frac{1}{a}e^{ax}\ln x - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx.$$

4.6. Интегралы, содержащие обратные тригонометрические функции.

$$\int \arcsin\frac{x}{a}\,dx = x\arcsin\frac{x}{a} + \sqrt{a^2 - x^2};$$

$$\int x\arcsin\frac{x}{a}\,dx = \left(\frac{x^2}{a} - \frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2};$$

$$\int x^2\arcsin\frac{x}{a}\,dx = \frac{x^3}{a}\arcsin\frac{x}{a} + \frac{1}{9}\left(x^2 + 2a^2\right)\sqrt{a^2 - x^2};$$

$$\int x^n\arcsin\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arcsin\frac{x}{a} - \frac{1}{n+1}\int \frac{x^{n+1}dx}{\sqrt{a^2 - x^2}};$$

$$\int \frac{1}{x}\arcsin\frac{x}{a}\,dx = \frac{x}{a} + \frac{1}{2\cdot 3\cdot 3}\frac{x^3}{a^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 5}\frac{x^5}{a^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 7}\frac{x^7}{a^7} + \ldots =$$

$$= \frac{x}{a} + \sum_{k=1}^{\infty} \frac{x^{2k+1}\prod_{i=1}^{k}\frac{(2i-1)}{2i}}{a^{2k+1}(2k+1)^2};$$

$$\int \frac{1}{x^2}\arcsin\frac{x}{a}\,dx = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|;$$

$$\int \frac{1}{x^n}\arcsin\frac{x}{a}\,dx = -\frac{1}{(n-1)x^{n-1}}\arcsin\frac{x}{a} + \frac{1}{n-1}\int \frac{dx}{x^{n-1}\sqrt{a^2 - x^2}} \quad (n \geqslant 2);$$

$$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2 - x^2}\,;$$

$$\int x\arccos\frac{x}{a}\,dx = \left(\frac{x^2}{a} - \frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2 - x^2}\,;$$

$$\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{1}{9}\left(x^2 + 2a^2\right)\sqrt{a^2 - x^2}\,;$$

$$\int x^n\arccos\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\arccos\frac{x}{a} + \frac{1}{n+1}\int\frac{x^{n+1}dx}{\sqrt{a^2 - x^2}};$$

$$\int \frac{1}{x}\arccos\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| - \frac{x}{a} - \frac{1}{2\cdot3\cdot3}\frac{x^3}{a^3} - \frac{1\cdot3}{2\cdot4\cdot5\cdot5}\frac{x^5}{a^5} -$$

$$-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot7}\frac{x^7}{a^7} - \ldots = \frac{\pi}{2}\ln|x| - \frac{x}{a} - \sum_{k=1}^{\infty}\frac{x^{2k+1}\prod\limits_{i=1}^{k}\frac{(2i-1)}{2i}}{a^{2k+1}\left(2k+1\right)^2};$$

$$\int \frac{1}{x^2}\arccos\frac{x}{a}\,dx = -\frac{1}{x}\arccos\frac{x}{a} + \frac{1}{a}\ln\frac{a + \sqrt{a^2 - x^2}}{x};$$

$$\int \frac{1}{x^n}\arccos\frac{x}{a}\,dx = -\frac{1}{(n-1)\,x^{n-1}}\arccos\frac{x}{a} - \frac{1}{n-1}\int\frac{dx}{x^{n-1}\sqrt{a^2 - x^2}};$$

$$\int \operatorname{arctg}\frac{x}{a}\,dx = x\operatorname{arctg}\frac{x}{a} - \frac{a}{2}\ln\left(a^2 + x^2\right);$$

$$\int x\operatorname{arctg}\frac{x}{a}\,dx = \frac{1}{2}\left(x^2 + a^2\right)\operatorname{arctg}\frac{x}{a} - \frac{ax}{2};$$

$$\int x^2\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arctg}\frac{x}{a} - \frac{ax^3}{6} + \frac{a^3}{6}\ln\left(x^2 + a^2\right);$$

$$\int x^n\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arctg}\frac{x}{a} - \frac{a}{n+1}\int\frac{x^{n+1}dx}{x^2 + a^2} \qquad \left(n \ne -1\right);$$

$$\int \frac{1}{x}\operatorname{arctg}\frac{x}{a}\,dx = \frac{x}{a} - \frac{x^3}{3\cdot3\cdot a^3} + \frac{x^5}{5\cdot5\cdot a^5} - \frac{x^7}{7\cdot7\cdot a^7} + \ldots = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)^2 a^{2k+1}} \qquad \left(|x| < |a|\right);$$

$$\int \frac{1}{x^2}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{1}{x}\operatorname{arctg}\frac{x}{a} - \frac{1}{2a}\ln\frac{x^2 + a^2}{x^2};$$

$$\int \frac{1}{x^n}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{1}{(n-1)\,x^{n-1}}\operatorname{arctg}\frac{x}{a} + \frac{a}{n-1}\int\frac{dx}{x^{n-1}\left(x^2 + a^2\right)} \qquad (n \ne 1);$$

$$\int \operatorname{arcctg}\frac{x}{a}\,dx = x\operatorname{arcctg}\frac{x}{a} + \frac{a}{2}\ln\left(x^2 + a^2\right);$$

$$\int x\operatorname{arcctg}\frac{x}{a}\,dx = \frac{1}{2}\left(x^2 + a^2\right)\operatorname{arcctg}\frac{x}{a} + \frac{ax}{2};$$

$$\int x^2\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arcctg}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln\left(x^2 + a^2\right);$$

$$\int x^n \operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^{n+1}}{n+1}\operatorname{arcctg}\frac{x}{a} + \frac{a}{n+1}\int\frac{x^{n+1}dx}{x^2+a^2} \quad (n \neq -1);$$

$$\int \frac{1}{x}\operatorname{arcctg}\frac{x}{a}\,dx = \frac{\pi}{2}\ln|x| - \frac{x}{a} + \frac{x^3}{3\cdot 3\cdot a^3} - \frac{x^5}{5\cdot 5\cdot a^5} + \frac{x^7}{7\cdot 7\cdot a^7} + \ldots =$$

$$= \frac{\pi}{2}\ln|x| + \sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^{2k+1}}{(2k+1)^2 a^{2k+1}} \quad (|x| < |a|);$$

$$\int \frac{1}{x^2}\operatorname{arcctg}\frac{x}{a}\,dx = -\frac{1}{x}\operatorname{arcctg}\frac{x}{a} + \frac{1}{2a}\ln\frac{x^2+a^2}{x^2};$$

$$\int \frac{1}{x^n}\operatorname{arcctg}\frac{x}{a}\,dx = \frac{-1}{(n-1)x^{n-1}}\operatorname{arcctg}\frac{x}{a} - \frac{a}{n-1}\int\frac{dx}{x^{n-1}(x^2+a^2)} \quad (n \neq 1).$$

4.7. Интегралы, содержащие гиперболические функции.

$$\int \operatorname{sh} ax\,dx = \frac{1}{a}\operatorname{ch} ax;$$

$$\int \operatorname{ch} ax\,dx = \frac{1}{a}\operatorname{sh} ax;$$

$$\int \operatorname{ch}^2 ax\,dx = \frac{1}{2a}\operatorname{sh} ax\operatorname{ch} ax + \frac{x}{2};$$

$$\int \operatorname{sh}^n ax\,dx = \frac{1}{an}\operatorname{sh}^{n-1} ax\operatorname{ch} ax - \frac{n-1}{n}\int\operatorname{sh}^{n-2} ax\,dx \quad (n > 0),$$

$$= \frac{1}{a(n-1)}\operatorname{sh}^{n+1} ax\operatorname{ch} ax - \frac{n+2}{n+1}\int\operatorname{sh}^{n+2} ax\,dx \quad (n < 0,\ n \neq -1);$$

$$\int \operatorname{ch}^n ax\,dx = \frac{1}{an}\operatorname{sh} ax\operatorname{ch}^{n-1} ax + \frac{n-1}{n}\int\operatorname{ch}^{n-2} ax\,dx \quad (n > 0),$$

$$= \frac{1}{a(n+1)}\operatorname{sh} ax\operatorname{ch}^{n+1} ax + \frac{n+2}{n+1}\int\operatorname{ch}^{n+2} ax\,dx \quad (n < 0,\ n \neq -1);$$

$$\int \frac{dx}{\operatorname{sh} ax} = \frac{1}{a}\ln\left|\operatorname{th}\frac{ax}{2}\right|;$$

$$\int \frac{dx}{\operatorname{ch} ax} = \frac{2}{a}\operatorname{arctg} e^{ax};$$

$$\int x\operatorname{sh} ax\,dx = \frac{1}{a}x\operatorname{ch} ax - \frac{1}{a^2}\operatorname{sh} ax;$$

$$\int x\operatorname{ch} ax\,dx = \frac{1}{a}x\operatorname{sh} ax - \frac{1}{a^2}\operatorname{ch} ax;$$

$$\int \operatorname{th} ax\,dx = \frac{1}{a}\ln\operatorname{ch} ax;$$

$$\int \operatorname{cth} ax\,dx = \frac{1}{a}\ln|\operatorname{sh} ax|;$$

$$\int \operatorname{th}^2 ax\,dx = x - \frac{\operatorname{th} ax}{a};$$

$$\int \operatorname{cth}^2 ax\,dx = x - \frac{\operatorname{cth} ax}{a};$$

$$\int \operatorname{sh} ax \operatorname{sh} bx\,dx = \frac{1}{a^2 - b^2}\left(a \operatorname{sh} bx \operatorname{ch} ax - b \operatorname{ch} bx \operatorname{sh} ax\right) \quad (a^2 \neq b^2);$$

$$\int \operatorname{ch} ax \operatorname{ch} bx\,dx = \frac{1}{a^2 - b^2}\left(a \operatorname{sh} ax \operatorname{ch} bx - b \operatorname{sh} bx \operatorname{ch} ax\right) \quad (a^2 \neq b^2);$$

$$\int \operatorname{ch} ax \operatorname{sh} bx\,dx = \frac{1}{a^2 - b^2}\left(a \operatorname{sh} ax \operatorname{sh} bx - b \operatorname{ch} bx \operatorname{ch} ax\right) \quad (a^2 \neq b^2);$$

$$\int \operatorname{sh} ax \sin ax\,dx = \frac{1}{2a}\left(\operatorname{ch} ax \sin ax - \operatorname{sh} ax \cos ax\right);$$

$$\int \operatorname{ch} ax \sin ax\,dx = \frac{1}{2a}\left(\operatorname{sh} ax \sin ax - \operatorname{ch} ax \cos ax\right);$$

$$\int \operatorname{sh} ax \cos ax\,dx = \frac{1}{2a}\left(\operatorname{ch} ax \cos ax + \operatorname{sh} ax \sin ax\right);$$

$$\int \operatorname{ch} ax \cos ax\,dx = \frac{1}{2a}\left(\operatorname{sh} ax \cos ax + \operatorname{ch} ax \sin ax\right).$$