CASE STUDY 2.4.1

Several years ago, a television program (inadvertently) spawned a conditional probability problem that led to more than a few heated discussions, even in the national media. The show was *Let's Make a Deal*, and the question involved the strategy that contestants should take to maximize their chances of winning prizes.

On the program, a contestant would be presented with three doors, behind one of which was the prize. After the contestant had selected a door, the host, Monty Hall, would open one of the other two doors, showing that the prize was not there. Then he would give the contestant a choice—either stay with the door initially selected or switch to the "third" door, which had not been opened.

For many viewers, common sense seemed to suggest that switching doors would make no difference. By assumption, the prize had a one-third chance of being behind each of the doors when the game began. Once a door was opened, it was argued that each of the remaining doors now had a one-half probability of hiding the prize, so contestants gained nothing by switching their bets.

Not so. An application of Definition 2.4.1 shows that it *did* make a difference—contestants, in fact, *doubled* their chances of winning by switching doors. To see why, consider a specific (but typical) case: The contestant has bet on Door #2 and Monty Hall has opened Door #3. Given that sequence of events, we need to calculate and compare the conditional probability of the prize being behind Door #1 and Door #2, respectively. If the former is larger (and we will prove that it is), the contestant should switch doors.

Table 2.4.1 shows the sample space associated with the scenario just described. If the prize is actually behind Door #1, the host has no choice but to open Door #3; similarly, if the prize is behind Door #3, the host has no choice but to open Door #1. In the event that the prize is behind Door #2, though, the host would (theoretically) open Door #1 half the time and Door #3 half the time.

Table 2.4.1

(Prize Location, Door Opened)	Probability
(1, 3)	1/3
(2, 1)	1/6
(2, 3)	1/6
(3, 1)	1/3

Notice that the four outcomes in S are not equally likely. There is necessarily a one-third probability that the prize is behind each of the three doors. However, the two choices that the host has when the prize is behind Door #2 necessitate that the two outcomes (2, 1) and (2, 3) share the one-third probability that represents the chances of the prize being behind Door #2. Each, then, has the one-sixth probability listed in Table 2.4.1.

Let A be the event that the prize is behind Door #2, and let B be the event that the host opened Door #3. Then

$$P(A|B) = P(\text{Contestant wins by not switching}) = [P(A \cap B)]/P(B) = \left[\tfrac{1}{6}\right]/\left[\tfrac{1}{3} + \tfrac{1}{6}\right] = \tfrac{1}{3}$$

Now, let A^* be the event that the prize is behind Door #1, and let B (as before) be the event that the host opens Door #3. In this case,

$$P(A^*|B) = P(\text{Contestant wins by switching}) = [P(A^* \cap B)]/P(B) = \left[\tfrac{1}{3}\right]/\left[\tfrac{1}{3} + \tfrac{1}{6}\right] = \tfrac{2}{3}$$

Common sense would have led us astray again! If given the choice, contestants should have *always* switched doors. Doing so upped their chances of winning from one-third to two-thirds.

Questions

2.4.1. Suppose that two fair dice are tossed. What is the probability that the sum equals 10 given that it exceeds 8?

2.4.2. Find $P(A \cap B)$ if $P(A) = 0.2$, $P(B) = 0.4$, and $P(A|B) + P(B|A) = 0.75$.

2.4.3. If $P(A|B) < P(A)$, show that $P(B|A) < P(B)$.

2.4.4. Let A and B be two events such that $P((A \cup B)^C) = 0.6$ and $P(A \cap B) = 0.1$. Let E be the event that either A or B but not both will occur. Find $P(E|A \cup B)$.

2.4.5. Suppose that in Example 2.4.2 we ignored the ages of the children and distinguished only *three* family types: (boy, boy), (girl, boy), and (girl, girl). Would the conditional probability of both children being boys given that at least one is a boy be different from the answer found on pp. 33–34? Explain.

2.4.6. Two events, A and B, are defined on a sample space S such that $P(A|B) = 0.6$, P(At least one of the events occurs) $= 0.8$, and P(Exactly one of the events occurs) $= 0.6$. Find $P(A)$ and $P(B)$.

2.4.7. An urn contains one red chip and one white chip. One chip is drawn at random. If the chip selected is red, that chip together with two additional red chips are put back into the urn. If a white chip is drawn, the chip is returned to the urn. Then a second chip is drawn. What is the probability that both selections are red?

2.4.8. Given that $P(A) = a$ and $P(B) = b$, show that

$$P(A|B) \geq \frac{a+b-1}{b}$$

2.4.9. An urn contains one white chip and a second chip that is equally likely to be white or black. A chip is drawn at random and returned to the urn. Then a second chip is drawn. What is the probability that a white appears on the second draw given that a white appeared on the first draw? (*Hint:* Let W_i be the event that a white chip is selected on the ith draw, $i = 1, 2$. Then $P(W_2|W_1) = \frac{P(W_1 \cap W_2)}{P(W_1)}$. If both chips in the urn are white, $P(W_1) = 1$; otherwise, $P(W_1) = \frac{1}{2}$.)

2.4.10. Suppose events A and B are such that $P(A \cap B) = 0.1$ and $P((A \cup B)^C) = 0.3$. If $P(A) = 0.2$, what does $P[(A \cap B)|(A \cup B)^C]$ equal? (*Hint:* Draw the Venn diagram.)

2.4.11. One hundred voters were asked their opinions of two candidates, A and B, running for mayor. Their responses to three questions are summarized below:

	Number Saying "Yes"
Do you like A?	65
Do you like B?	55
Do you like both?	25

(a) What is the probability that someone likes neither?

(b) What is the probability that someone likes exactly one?

(c) What is the probability that someone likes at least one?

(d) What is the probability that someone likes at most one?

(e) What is the probability that someone likes exactly one given that he or she likes at least one?