§11.1 Sequences

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Defn: A sequence is an infinite list of numbers.

$$\{a_1, a_2, a_3, \ldots\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$$

Some sequences are given by formulas:

Ex: $a_n = \frac{1}{2^n}$ i.e. $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$$

Ex: $a_n = \sqrt{2}$ $\quad \{\sqrt{2}\}_{n=1}^{\infty}$

$\{\sqrt{2}, \sqrt{2}, \sqrt{2}, \ldots\}$

Ex: Recursive Formula

$a_1 = 1 \quad a_2 = 1 \quad a_{n+1} = a_n + a_{n-1}$

$$\underbrace{\{1, 1, 2, 3, 5, 8, 13, \ldots\}}_{\text{Fibonacci Sequence}}$$

Getting formula for a sequence from the first few terms

Ex: $\left\{\frac{3}{5}, \frac{4}{25}, \frac{5}{125}, \frac{6}{625}, \ldots\right\}$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \ldots$

numerator is $n+2$ for $n \geq 1$

denominator is 5^n

$$a_n = \frac{n+2}{5^n}$$

Ex: $\left\{\frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \frac{1}{6}, \ldots\right\}$

denom: $n+1$

numerator: $(-1)^{n+1}$

$$a_n = \frac{(-1)^{n+1}}{n+1}$$

$$\left\{\frac{(-1)^{n+1}}{n+1}\right\}_{n=1}^{\infty} \quad \text{or} \quad \left\{\frac{(-1)^{n}}{n}\right\}_{n=2}^{\infty}$$

Convergence of Sequences

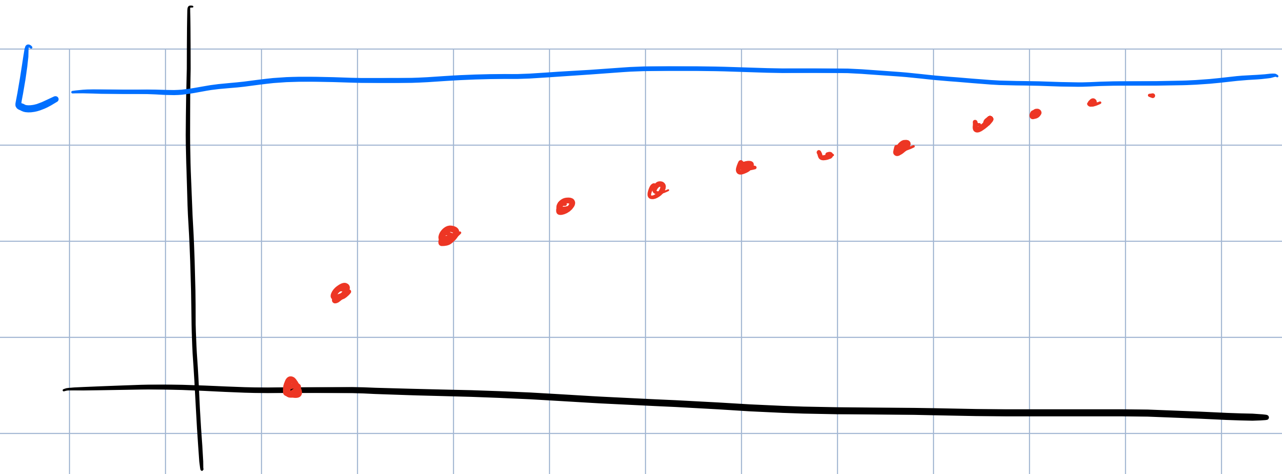
Similar to functions, we ask:

$$\lim_{n \to \infty} a_n = ?$$

$\{\ \}$

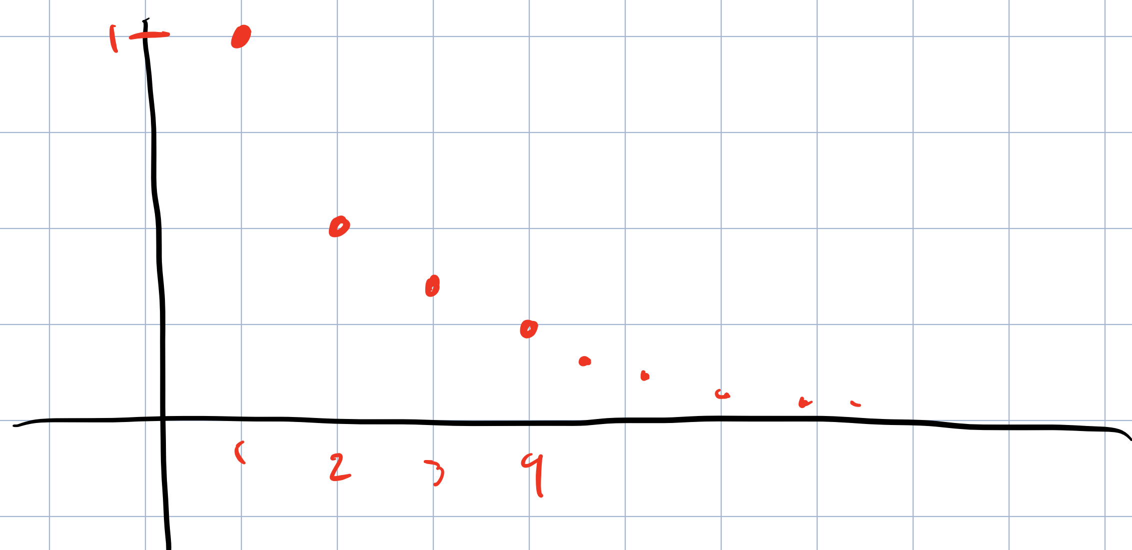
for some sequence $\{a_n\}$

Def'n: A sequence $\{a_n\}$ converges to L if we can make a_n arbitrarily close to L by making n large enough.

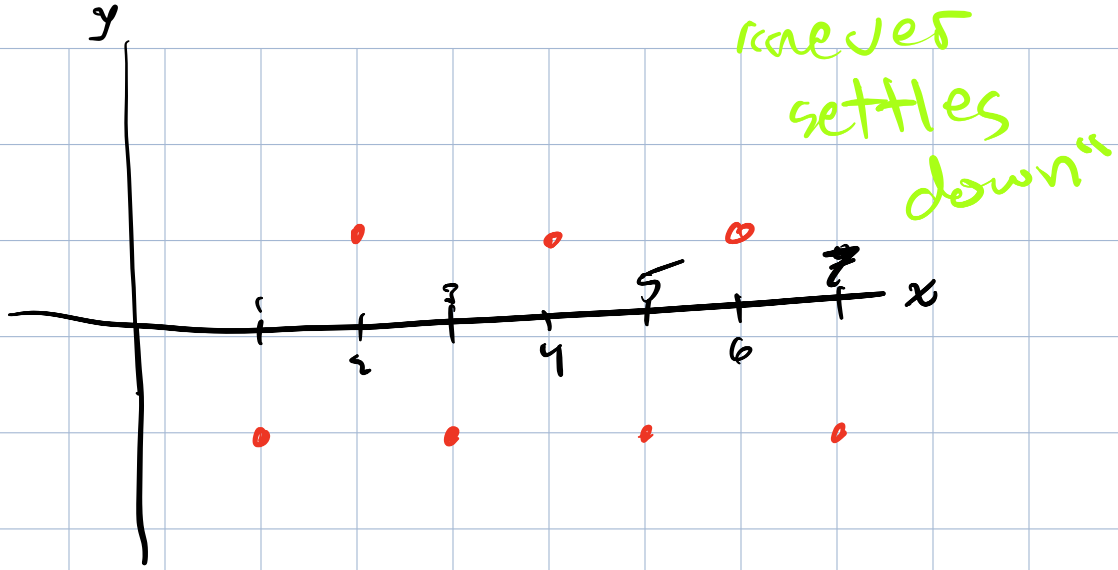


If $\{a_n\}$ converges to L we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$ as $n \to \infty$. If no such L exists, we say $\{a_n\}$ diverges.

<u>Ex</u>: $\lim_{n\to\infty} \frac{1}{n} = 0$



<u>Ex</u>: $\{(-1)^n\}_{n=1}^{\infty}$ is divergent



Ex: $\lim_{n\to\infty} n$ diverges

Theorem: If $\lim_{x\to\infty} f(x) = L$ for some function f and $a_n = f(n)$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n = L$.

Ex: $\lim_{n\to\infty} \frac{1}{n} = ?$

Notice: $f(x) = \frac{1}{x}$ has value $\frac{1}{n} = a_n$ when evaluated at n. We know $\lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{n\to\infty} \frac{1}{n} = 0 \Longleftarrow$

Ex: $\left\{\frac{\sin(n)}{n}\right\}_{n=1}^{\infty}$ Consider $f(x) = \frac{\sin(x)}{x}$

"Sandwich Theorem"

$$-1 \leq \sin(x) \leq 1$$

so

$$\underset{\substack{\downarrow \\ 0}}{-\frac{1}{x}} \leq \underset{\substack{\downarrow \\ 0}}{\frac{\sin(x)}{x}} \leq \underset{\substack{\downarrow \\ 0}}{\frac{1}{x}}$$

Theorem $\Longrightarrow$ $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

Limit Laws: If $\{a_n\}$ and $\{b_n\}$ are sequences with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ then:

(i) $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$

(ii) $\lim_{n\to\infty} \overset{\text{const.}}{c} \cdot a_n = c \lim_{n\to\infty} a_n = cL$

(iii) $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$

(iv) If $M \neq 0$ then

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{M}$$

Power Law: If $a_n > 0$ for all n and $p > 0$ then

$$\lim_{n\to\infty} (a_n)^p = L^p$$

Squeeze Theorem If $a_n \le b_n \le c_n$ for all n and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n = L$.

Absolute Value Theorem If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

$$\underset{>0}{3}$$

Ex: $\lim_{n\to\infty} \frac{1}{n^3} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^3 = \left(\lim_{n\to\infty} \frac{1}{n}\right)^3 = 0^3 = 0$

>0

Ex: $\lim_{n\to\infty} \left(\frac{(-1)^n}{n^3}\right)$

Notice: $\left|\frac{(-1)^n}{n^3}\right| = \frac{1}{n^3}$ and $\lim_{n\to\infty} \frac{1}{n^3} = 0$

So $\lim_{n\to\infty} \frac{(-1)^n}{n^3} = 0$ (by Abs. Val. Theorem)

??

$$\lim_{n\to\infty} \left(\frac{(-1)^n}{n^3}\right) \stackrel{?}{=} \lim_{n\to\infty} \left((-1)^n\right) \cdot \lim_{n\to\infty} \left(\frac{1}{n^3}\right)$$

divergent so we can't use Limit Law (iii)

Ex: Consider $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \left(\cancel{\frac{n}{n}}^{1} \cdot \frac{1}{1+\frac{1}{n}}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right) = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)} \quad \text{(iv)}$$

$$= \frac{1}{\lim_{n\to\infty}(1) + \lim_{n\to\infty}\frac{1}{n}} \quad \text{(i)}$$

$$= \frac{1}{1+0} = 1.$$

<u>Thm:</u> If $\lim_{n\to\infty} a_n = L$ and a function $y = f(x)$ is cont. at L then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

<u>Ex:</u> $a_n = \sin\left(\frac{\pi}{n}\right)$

Notice $\lim_{n\to\infty} \frac{\pi}{n} = \pi \lim_{n\to\infty} \frac{1}{n}$ (i:)

$$= 0$$

So $\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \sin(0)$

<u>Recall:</u> If $f(x) = b^x$ for $b \geq 0$ then

$$\lim_{x\to\infty} b^x = \begin{cases} \text{diverges} & b>1 \\ 1 & b=1 \\ 0 & 0\le b<1 \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} b^n = 0 \text{ if } 0\le b<1$$

But what about $b<0$?

Use the abs. val. thm.:

if $-1<b\le 0$ then

$$\lim_{n\to\infty} |b^n| = \lim_{n\to\infty} |b|^n = 0$$

$$\Rightarrow \lim_{n\to\infty} b^n = 0 \text{ when } -1<b\le 0$$

Monotonic Sequences

Defn: A sequence $\{a_n\}$ is

(i) increasing if $a_n < a_{n+1}$

for all $n \geq 1$

(ii) <u>decreasing</u> if $a_n \geq a_{n+1}$ for all $n \geq 1$.

If $\{a_n\}$ either is decreasing or increasing, it is <u>monotonic</u>.

<u>Defn</u>: A sequence $\{a_n\}$ is called bounded if there are M, M' s.t.

$$M' \leq a_n \leq M \quad \text{for all } n \geq 1.$$

<u>Thm</u>: A monotonic sequence which is bounded is convergent.