[MAA 3.7]

TRIGONOMETRIC FUNCTIONS

SOLUTIONS

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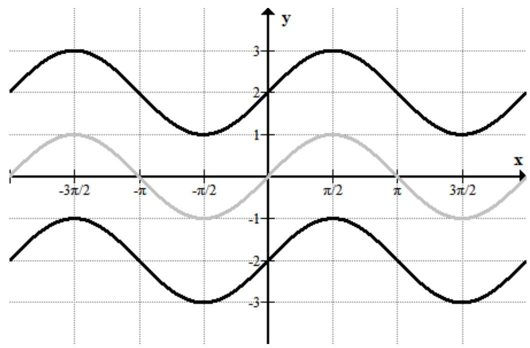
O. Practice questions

1. (a)

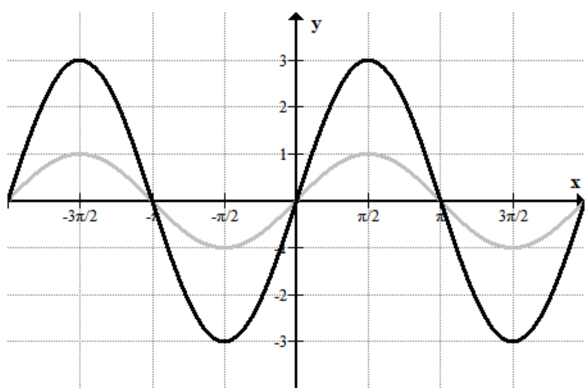


(b)

Function	Amplitude	Period	Central axis	Range
$y=\sin x$	1	2π	$y=0$	$-1\le y\le 1$
$f(x)=\sin x+2$	1	2π	$y=2$	$1\le y\le 3$
$g(x)=\sin x-2$	1	2π	$y=-2$	$-3\le y\le -1$

(c) $y=c$

2. (a)



(b)

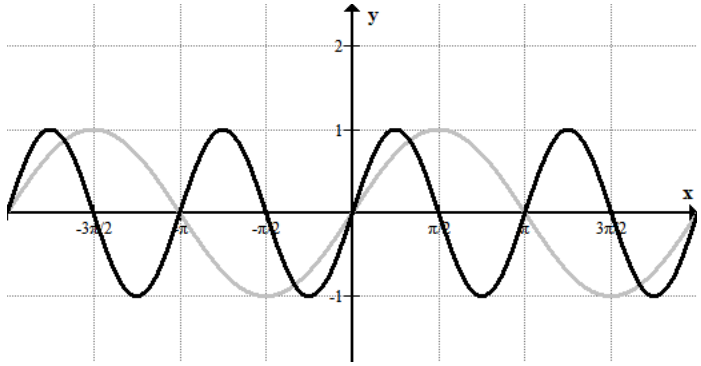
Function	Amplitude	Period	Central axis	Range
$y=\sin x$	1	2π	$y=0$	$-1\le y\le 1$
$f(x)=3\sin x$	3	2π	$y=0$	$-3\le y\le 3$

(c) 3

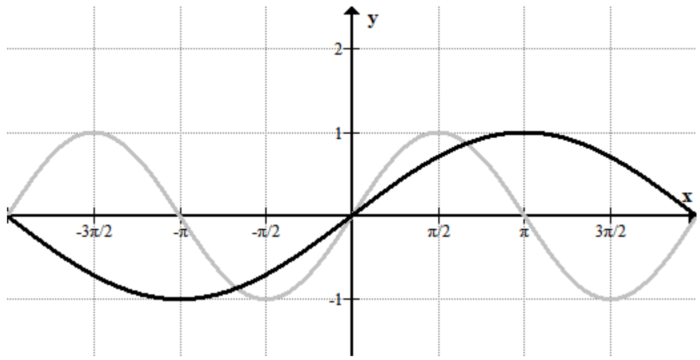
(d) $|a|$

(e) (i) $|a|$ (ii) $y=c$

3. (a)



(b)



(c)

Function	Amplitude	Period	Central axis	Range
$y = \sin x$	1	2π	$y = 0$	$-1 \le y \le 1$
$f(x) = \sin 2x$	1	π	$y = 0$	$-1 \le y \le 1$
$g(x) = \sin \dfrac{x}{2}$	1	4π	$y = 0$	$-1 \le y \le 1$

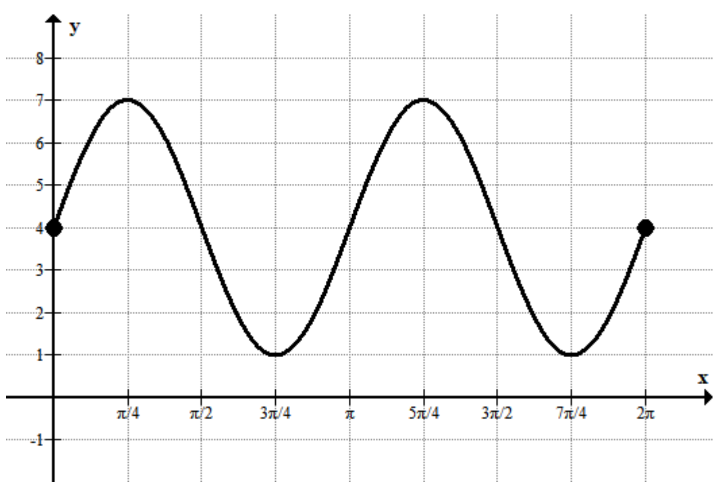
(d) $\dfrac{2\pi}{b}$

(e) (i) $|a|$ (ii) $y = c$ (iii) $\dfrac{2\pi}{b}$

4. (a)

Function	Amplitude	Period	Central axis	Range
$f(x)$	3	π	$y = 4$	$1 \le y \le 7$

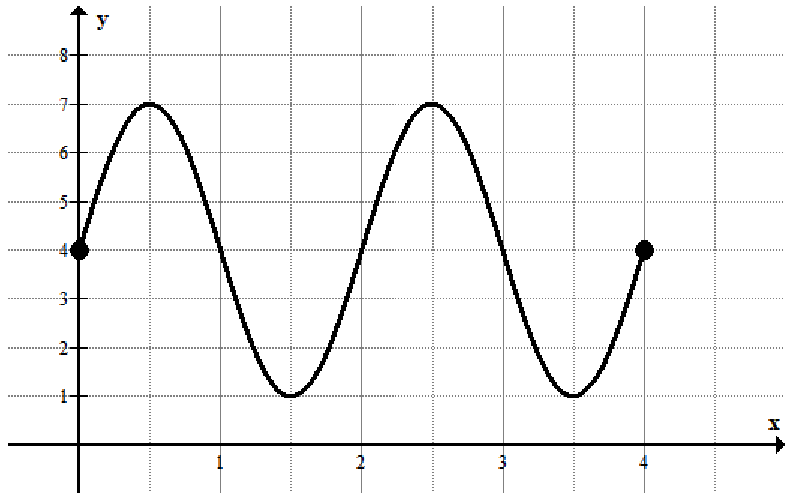
(b)



5. (a)

Function	Amplitude	Period	Central axis	Range
$f(x)$	3	2	$y = 4$	$1 \le y \le 7$

(b)



6. (a)

Function	Amplitude	Period	Central axis	Range
$f(x)$	a	$\frac{2\pi}{b}$	$y = c$	$c - a \le y \le c + a$

(b)

Function	Period	Central axis	Range
$f(x)$	$\frac{\pi}{b}$	$y = c$	$y \in \mathbb{R}$

7.

Function	Amplitude	Period	Central axis	Range
$f(x) = \sin x$	1	2π	$y = 0$	$-1 \le y \le 1$
$f(x) = \cos x$	1	2π	$y = 0$	$-1 \le y \le 1$
$f(x) = \sin x + 1$	1	2π	$y = 1$	$0 \le y \le 2$
$f(x) = \sin x - 1$	1	2π	$y = -1$	$-2 \le y \le 0$
$f(x) = 5\sin x$	5	2π	$y = 0$	$-5 \le y \le 5$
$f(x) = -7\sin x$	7	2π	$y = 0$	$-7 \le y \le 7$
$f(x) = \sin 4x$	1	$\pi/2$	$y = 0$	$-1 \le y \le 1$
$f(x) = -\cos 4x$	1	$\pi/2$	$y = 0$	$-1 \le y \le 1$
$f(x) = 3\sin 4x$	3	$\pi/2$	$y = 0$	$-3 \le y \le 3$
$f(x) = 3\sin 4x + 10$	3	$\pi/2$	$y = 10$	$7 \le y \le 13$
$f(x) = 3\sin 4x - 2$	3	$\pi/2$	$y = -2$	$-5 \le y \le 1$
$f(x) = -5\sin 3x$	5	$2\pi/3$	$y = 0$	$-1 \le y \le 1$
$f(x) = -5\sin x + 10$	5	2π	$y = 10$	$5 \le y \le 15$
$f(x) = \tan x$		π	$y = 0$	$y \in R$
$f(x) = \tan 4x$		$\pi/4$	$y = 0$	$y \in R$
$f(x) = 5\tan 4x + 10$		$\pi/4$	$y = 10$	$y \in R$

8. (a) amplitude = 80, central value = 100, period = π/2

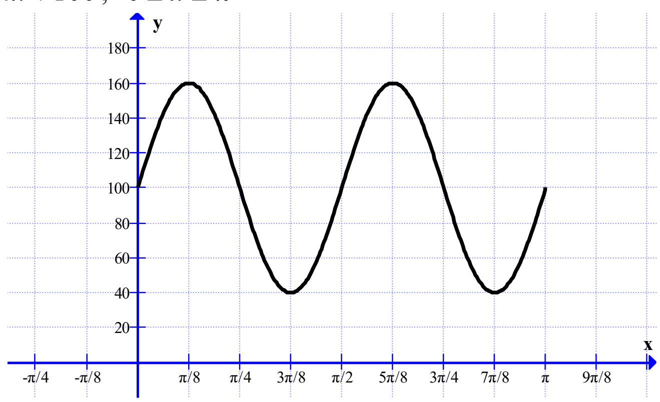
(b) $f(x) = 80\sin 4x + 100$, since $B = \dfrac{2\pi}{Period} = \dfrac{2\pi}{\pi/2} = 4$

(c) (i) $f(x) = -80\sin 4(x - \frac{\pi}{4}) + 100$, ($D = \frac{\pi}{4}$ is the position of the 2nd (↓) root)

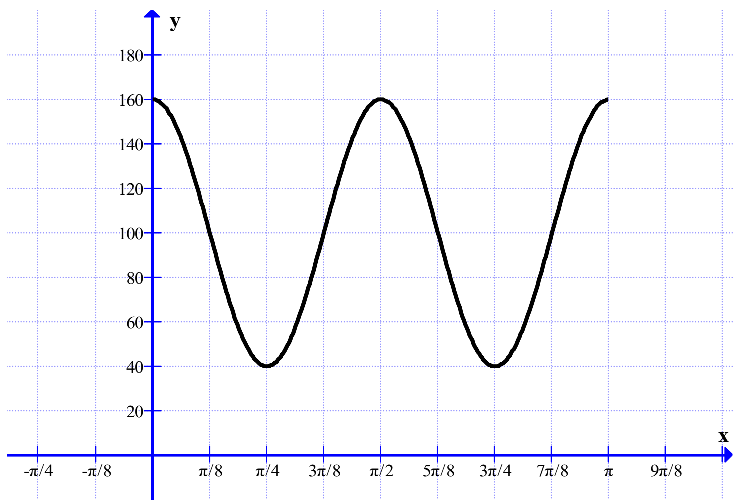
(ii) $f(x) = 80\cos 4(x - \frac{\pi}{8}) + 100$, ($D = \frac{\pi}{8}$ is the position of the maximum)

(iii) $f(x) = -80\cos 4(x - \frac{3\pi}{8}) + 100$ ($D = \frac{3\pi}{8}$ is the position of the minimum)

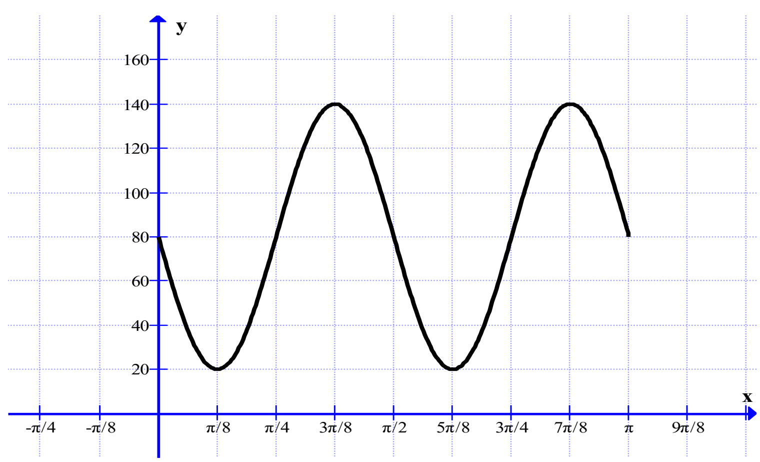
9. $f(x) = 60\sin 4x + 100$, $0 \le x \le \pi$



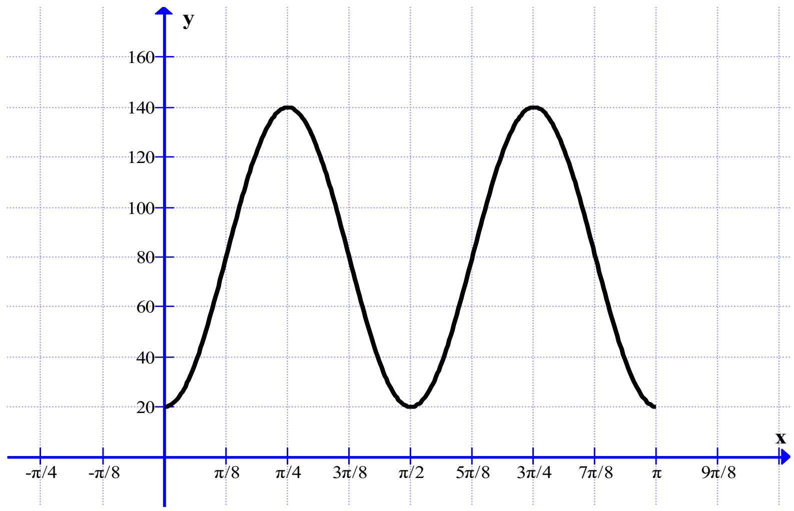
10. $f(x) = 60\cos 4x + 100$, $0 \le x \le \pi$



11. $f(x) = -60\sin 4x + 80$, $0 \le x \le \pi$



12. $f(x) = -60\cos 4x + 80,\ \ 0 \le x \le \pi$



(i) $k = 140$ (ii) $k = 20$ (iii) $20 < k < 140$ (iv) $k < 20$ or $k > 140$

A. Exam style questions (SHORT)

13. (a) $f(x) = 15 \sin (6x)$

(b) period = $2\pi/6 = \pi/3$

(c) $15 \sin 6x = 0$, (OR $\sin 3x = 0$ **and** $\cos 3x = 0$)

$6x = 0, \pi, 2\pi$

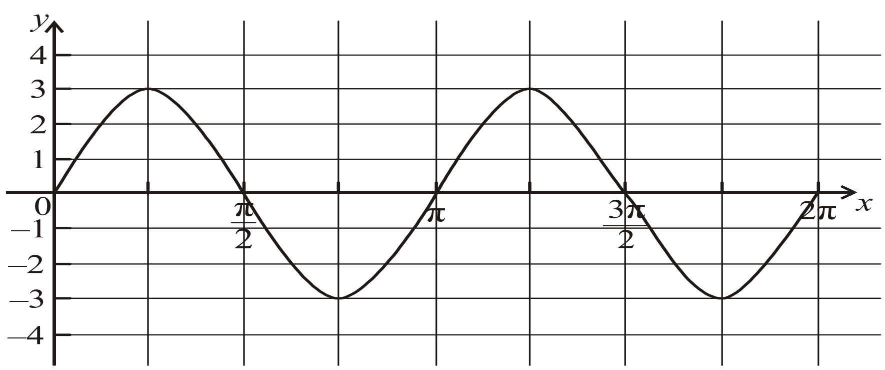
$x = 0, \dfrac{\pi}{6}, \dfrac{\pi}{3}$

14. From sketch of graph $y = 4\sin\left(3x + \dfrac{\pi}{2}\right)$ or by observing $|\sin \theta| \le 1$.

$k > 4, k < -4$

15. (a) period = π

(b)



(c) 4 (solutions)

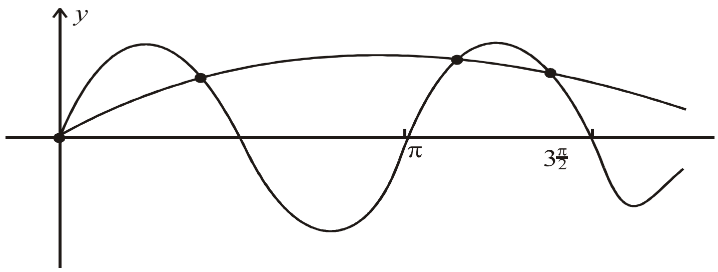
16. (a) $p = 30$

(b) Period = $\dfrac{2\pi}{q} = \dfrac{\pi}{2} \Rightarrow q = 4$

17. (a) (i) -1

(ii) 4π (accept 720°)

(b)



number of solutions: 4

18. $3 = p + q\cos 0 \;\Rightarrow\; 3 = p + q$

$-1 = p + q\cos\pi \Rightarrow -1 = p - q$

(i) $p = 1$ (ii) $q = 2$

19. (a) $h(x) = 4\cos\left(\dfrac{\frac{3x}{2}+1}{3}\right) - 1 = 4\cos\left(\dfrac{1}{2}x + \dfrac{1}{3}\right) - 1$

(b) period is 4π

(c) range is $-5 \le h(x) \le 3$ $([-5, 3])$

20. (a) (i) amplitude $= \dfrac{7+3}{2} = 5 \;\Rightarrow\; p = -5$

(ii) period $= 8 \;\Rightarrow\; q = 0.785 \left(= \dfrac{2\pi}{8} = \dfrac{\pi}{4}\right)$

(iii) $r = \dfrac{7-3}{2} \;\Rightarrow r = 2$

(b) $k = -3$ (accept $y = -3$)

21. **METHOD 1**

The value of cosine varies between –1 and +1. Therefore:

$t = 0 \Rightarrow a + b = 14.3$

$t = 6 \Rightarrow a - b = 10.3$

$\Rightarrow a = 12.3 \quad b = 2$

Period = 12 hours $\Rightarrow \dfrac{2\pi(12)}{k} = 2\pi \Rightarrow k = 12$

METHOD 2

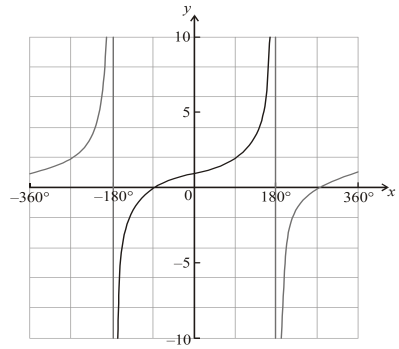
From graph: Midpoint $= a = 12.3$

Amplitude $= b = 2$

Period $= \dfrac{2\pi}{\frac{2\pi}{k}} = 12 \Rightarrow k = 12$

22. $a = 4, b = 2, c = \frac{\pi}{2}\left(\text{or } \frac{3\pi}{2} etc\right)$

23. (a)



(b) (i) Period = 360° (accept 2π) (ii) $f(90°) = 2$

(c) 270°, –90°

24.

Recognition of stretch or compression parallel to x-axis

Scale factor is $\frac{6}{\pi}$ or $\frac{\pi}{6}$ respectively

Reflection in x-axis

Recognition of stretch parallel to y-axis with scale factor 2

Recognition of translation $\begin{pmatrix} 0 \\ 8 \end{pmatrix}$

A correct sequence (i.e. the translation must be stated last).

25.

Since range goes from -4 to $2 \Rightarrow a = 3$

Since curve is shifted right by $\frac{\pi}{4}, \Rightarrow b = -\frac{\pi}{4}$

Since curve has been shifted in vertical by one unit down $\Rightarrow c = -1$

$a = 3 \quad b = -\frac{\pi}{4} \quad c = -1$

26. (a) [-4, 2]

(b) 4 solutions

(c) (i) exactly 2 solutions if $k = 2$ or $k = -4$

(ii) exactly 4 solutions if $-4 < k < 2$

(iii) no solutions if $k > 2$ or $k < -4$.

27.

(a) Either finding depths graphically, using $\sin\frac{\pi t}{6} = \pm 1$ or solving $h'(t) = 0$ for t

$h(t)_{\max} = 12$ (m), $h(t)_{\min} = 4$ (m)

(b) Attempting to solve $8 + 4\sin\frac{\pi t}{6} = 8$ algebraically or graphically

$t \in [0, 6] \cup [12, 18] \cup \{24\}$

28. **With GDC**

(a) $x = 2.79$

(b) $x = 3.32$ or $x = 5.41$

Without GDC

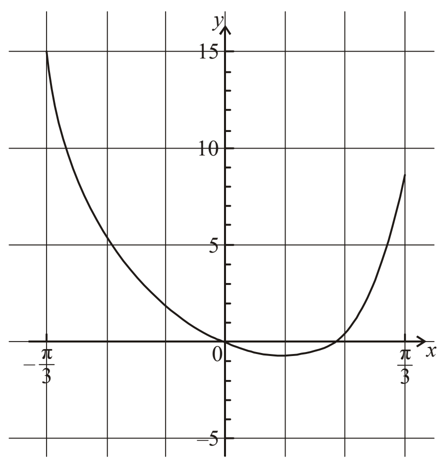
(a) $\sin(x + \frac{\pi}{9}) = 0 \Leftrightarrow x + \frac{\pi}{9} = k\pi \Rightarrow \Leftrightarrow x = -\frac{\pi}{9} + k\pi$ gives $x = \frac{8\pi}{9}$ $(\equiv 2.79)$

(b) $\sin\left(x + \frac{\pi}{9}\right) = -\frac{1}{2} \Rightarrow$

$$x + \frac{\pi}{9} = -\frac{\pi}{6} + 2k\pi \Rightarrow x = -\frac{\pi}{6} - \frac{\pi}{9} + 2k\pi \Rightarrow x = \frac{-5\pi}{18} + 2k\pi, \quad \text{gives } x = \frac{31\pi}{18} \quad (\equiv 5.41)$$

$$x + \frac{\pi}{9} = -\frac{5\pi}{6} + 2k\pi \Rightarrow x = -\frac{5\pi}{6} - \frac{\pi}{9} + 2k\pi \Rightarrow x = \frac{-17\pi}{18} + 2k\pi, \quad \text{gives } x = \frac{19\pi}{18} \quad (\equiv 3.32)$$

29. (a)

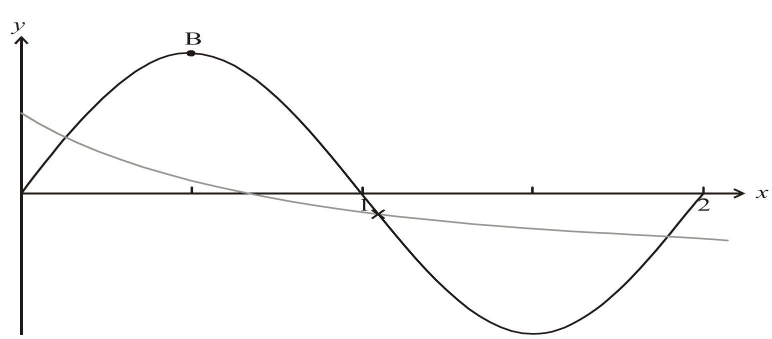


passing through (0, 0), range approximately –1 to 15.

(b) $x = -0.207$ $x = 0.772$

30. (a) $b = 6$

(b)



(c) $x = 1.05$ (no additional solutions)

31. (a) $y = \sin x \;\rightarrow\; y = 2\sin x \;\rightarrow\; y = 2\sin(x-1) \;\rightarrow\; y = 2\sin(3x-1)$

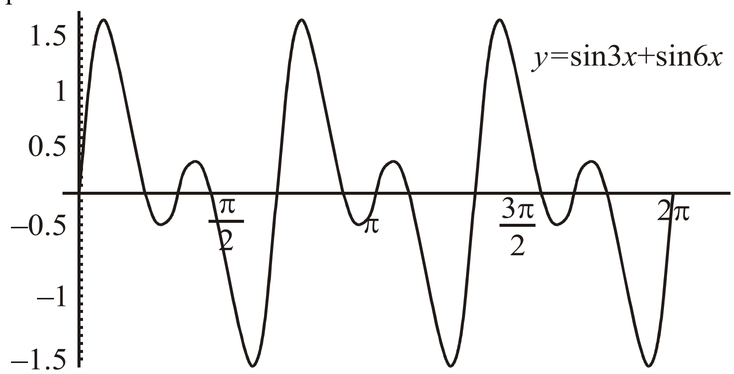
Vertical stretch scale factor 2

Horizontal translation 1 unit to the right

Horizontal stretch scale factor 1/3 (i.e. shrink)

(b) $-1.36 \le x \le -0.832$ or $0.354 \le x \le 1.14$.

32. The graph looks like



(a) [-1.76, 1.76]

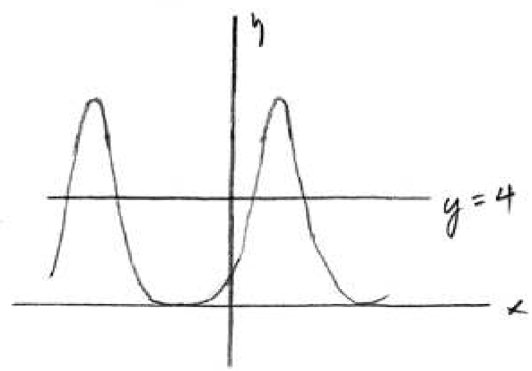
(b) 11 solutions

(c) Period = $\dfrac{2\pi}{3}$

33. (a) $(f \circ g)(x) = \mathrm{e}^{2\sin\left(\frac{\pi x}{2}\right)}$

Period = period of $\sin\left(\dfrac{\pi x}{2}\right) = \dfrac{2\pi}{\frac{\pi}{2}} = 4$

(b)



The first positive interval is (0.488, 1.51)

Then we add 4 (period) to obtain the next interval.

$x \in (0.488 + 4k, 1.51 + 4k)\ (k \in \mathbb{Z})$

B. Exam style questions (LONG)

34. (a) $f(1) = 3$ $f(5) = 3$

(b) **EITHER** distance between successive maxima = period = 5 – 1 = 4

OR period = $\frac{2\pi}{\frac{\pi}{2}} = 4$

(c) Amplitude = $A = \frac{3-(-1)}{2} = \frac{4}{2} = 2$

Midpoint value = $B = \frac{3+(-1)}{2} = \frac{2}{2} = 1$

(d) $f(x) = 2 \Rightarrow 2\sin\left(\frac{\pi}{2}x\right) + 1 = 2 \Rightarrow x = \frac{1}{3}$ or $\frac{5}{3}$ or $\frac{13}{3}$

(e) (i) $k = -1$ (ii) $1 \le k < 3$ (iii) $-1 < k < 1$ or $k = 3$ (iv) $k < -1$ or $k > 3$

35. (a) (i) $10 + 4\sin 1 = 13.4$

(ii) At 2100, $t = 21$

$10 + 4\sin 10.5 = 6.48$

(b) (i) 14 metres

(ii) $14 = 10 + 4\sin\left(\frac{t}{2}\right) \Rightarrow t = \pi$ (=3.14)

(c) (i) 4

(ii) $10 + 4\sin\left(\frac{t}{2}\right) = 7 \Rightarrow t = 7.98$

(iii) depth < 7 from 8 –11 = 3 hours, from 2030 – 2330 = 3 hours

therefore, total = 6 hours

36. (a) (i) $Q = \frac{1}{2}(14.6 - 8.2) = 3.2$ (ii) $P = \frac{1}{2}(14.6 + 8.2) = 11.4$

(b) $10 = 11.4 + 3.2\cos\left(\frac{\pi}{6}t\right)$

$t = 3.8648$. $t = 3.86$(3 s.f.)

(c) (i) By symmetry, next time is 12 – 3.86... = 8.135... $t = 8.14$ (3 s.f.)

(ii) From above, first interval is $3.86 < t < 8.14$

This will happen again, 12 hours later, so $15.9 < t < 20.1$

37. (a) (i) 7 (ii) 1 (iii) 10

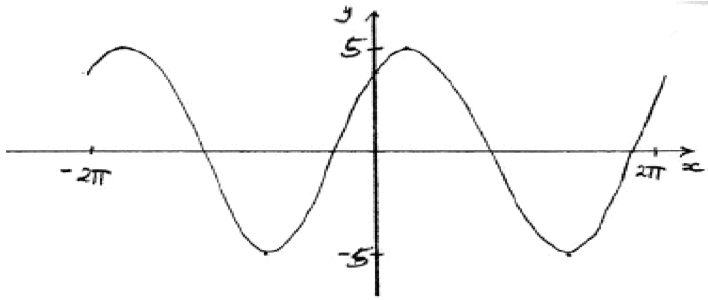
(b) (i) $A = \frac{18-2}{2} = 8$

(ii) $C = 10$

(iii) period = 12 $B = \frac{\pi}{6}$

(c) $t = 3.52$, $t = 10.5$, between 03:31 and 10:29 (accept 10:30)

38. (a)



(b) (i) 5 (ii) 2π (6.28) (iii) –0.927

(c) $f(x) = 5 \sin (x + 0.927)$ (accept $p = 5$, $q = 1$, $r = 0.927$)

(d) 3 s.f. values which round to –5.6, 0.64

(e) $k = –5$, $k = 5$

39. (a) When $t = 1$, $l = 33 + 5 \cos 720 = 38$

(b) $l_{\min} = 33 – 5 = 28$

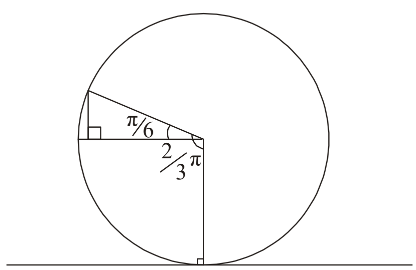
(c) $33 = 33 + 5\cos720t \Rightarrow t = 1/8$

(d) period = $\frac{360}{720}\left(=\frac{1}{2}\right)$

40. (a) arc AB = $r\theta = 7.85$ (m)

(b) Area of sector AOB $A = \frac{1}{2}r^2\theta = 58.9$ (m^2)

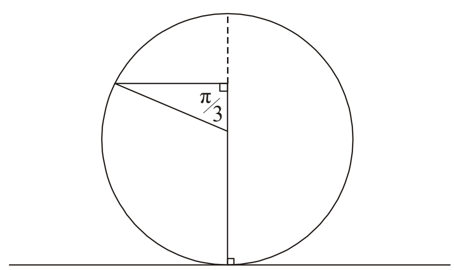
(c) **METHOD 1**



angle = $\frac{\pi}{6}$ (30°)

height = $15 + 15 \sin \frac{\pi}{6} = 22.5$ (m)

METHOD 2



angle = $\frac{\pi}{3}$ (60°)

height = $15 + 15 \cos \frac{\pi}{3} = 22.5$ (m)

(d) (i) $h\left(\frac{\pi}{4}\right) = 15 - 15\cos\left(\frac{\pi}{2}+\frac{\pi}{4}\right) = 25.6$ (m)

(ii) $h(0) = 15 - 15 \cos\left(0+\frac{\pi}{4}\right) = 4.39$(m)

(iii) **METHOD 1**

Highest point when $h = 30$

$30 = 15 - 15 \cos\left(2t+\frac{\pi}{4}\right) \Leftrightarrow t = 1.18$ $\left(\text{accept } \frac{3\pi}{8}\right)$

METHOD 2

Using graph $t = 1.18$

41. (a) (i) 100 (metres)

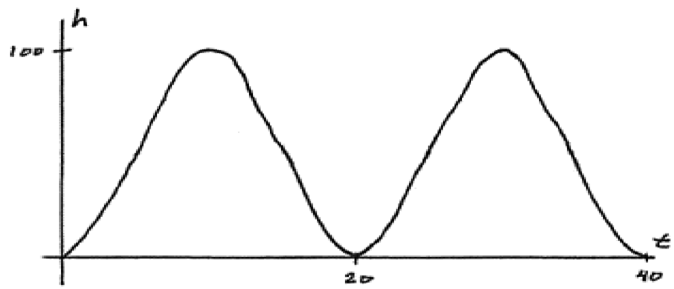
(ii) 50 (metres)

(b) (i) Symmetry with $h(2) = 9.5$

$h(8) = 100 - 9.5 = 90.5$

(ii) $h(21) = h(1) = 2.4$

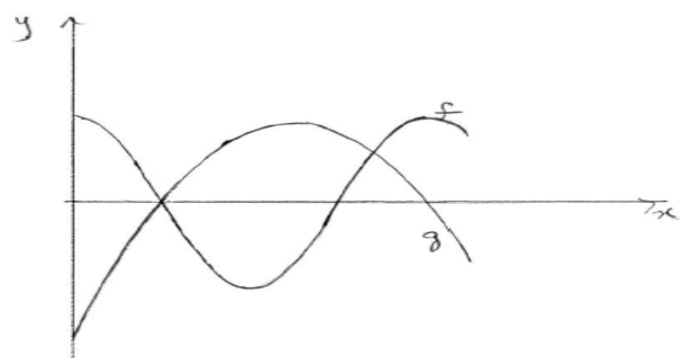
(c)



(d) $b = \frac{2\pi}{20}\left(= \frac{\pi}{10}\right)$ (accept b =18 if working in degrees)

$a = -50$, $c = 50$

42. (a)



(b) (i) (2,0) (or $x = 2$)

(ii) period = 8

(iii) amplitude = 5

(c) (i) (2, 0), (8, 0) (or $x = 2$, $x = 8$)

(ii) $x = 5$ (must be an equation)

(d) intersect when $x = 2$ and $x = 6.79$

43. (a) $\Delta = 0 \Leftrightarrow k^2 - 4 \times 4 \times 1 = 0 \Leftrightarrow k^2 = 16 \Leftrightarrow k = 4, k = -4$

(b) using $\cos 2\theta = 2\cos^2\theta - 1$, $f(\theta) = 4\cos^2\theta + 4\cos\theta + 1$

(c) (i) 1

(ii) **METHOD 1**

Solve for $\cos\theta$

$\cos\theta = -\frac{1}{2}$, $\theta = 240, 120, -240, -120$

METHOD 2

Directly by GDC: $\theta = 240, 120, -240, -120$

(d) Using graph, $c = 9$