PROJECTILE MOTION *Optimum angle*

The objective is to find the angle of launch that will maximize the range. Once we have this angle, we will find several other aspects of this trajectory. Start with the general range equation, developed elsewhere:

$$R = \frac{v_0}{g}\cos(\theta)\left[v_0 \sin(\theta) + \sqrt{\left(v_0 \sin(\theta)\right)^2 + 2\,g\,y_0}\right] \tag{1}$$

To find the angle that will maximize this, we differentiate with respect to the angle, leaving the other parameters fixed. The derivative is

$$v_0\left(v_0 \sin(\theta) + \sqrt{v_0^{\,2} \sin(\theta)^2 + 2\,g\,y_0}\right)\frac{\left(v_0 \cos(\theta)^2 - \sin(\theta)\sqrt{v_0^{\,2}\sin(\theta)^2 + 2\,g\,y_0}\right)}{g\sqrt{v_0^{\,2}\sin(\theta)^2 + 2\,g\,y_0}} = 0$$

The quantity in the first brackets cannot be zero, so

$$v_0 \cos(\theta)^2 - \sin(\theta)\sqrt{v_0^{\,2}\sin(\theta)^2 + 2\,g\,y_0} = 0 \qquad \text{from which} \qquad \frac{\cos(\theta)^4}{\sin(\theta)^2} - \sin(\theta)^2 = \frac{2\,g\,y_0}{v_0^{\,2}}$$

Using some trig identities (Tuma, p61), get everything in terms of one function; here, the tangent:

$$\frac{\dfrac{1}{\left(1+\tan(\theta)^2\right)^2}}{\dfrac{\tan(\theta)^2}{1+\tan(\theta)^2}} - \frac{\tan(\theta)^2}{1+\tan(\theta)^2} = \frac{2\,g\,y_0}{v_0^{\,2}}$$

and then solve this for the angle. We get

$$\boxed{\theta_{opt} = \operatorname{atan}\left(\frac{v_0}{\sqrt{v_0^{\,2} + 2\,g\,y_0}}\right) = \operatorname{atan}\left(\frac{v_0}{v_f}\right)} \tag{2}$$

Observe that this is the arctangent of the initial velocity divided by the final velocity. This says that we have a right triangle, with the initial and final velocities perpendicular to each other (for this optimum angle), so that we have a hypotenuse of

$$\sqrt{2\left(v_0^{\,2} + g\,y_0\right)}$$

With this we can define some alternative versions of the optimum angle:

$$\theta_{opt} = \operatorname{asin}\left[\frac{v_0}{\sqrt{2\left(v_0^2 + g_{y.0}\right)}}\right] = \operatorname{asin}\left[\frac{1}{\sqrt{2\left(1 + \frac{g\,y_0}{v_0^2}\right)}}\right] \qquad \theta_{opt} = \operatorname{acos}\left[\frac{\sqrt{v_0^2 + 2\,g\,y_0}}{\sqrt{2\left(v_0^2 + g\,y_0\right)}}\right]$$

$$\theta_{opt} = \frac{\pi}{2} - \operatorname{atan}\left(\frac{v_f}{v_0}\right) = \frac{\pi}{2} - \operatorname{atan}\left(\sqrt{1 + \frac{2\,g\,y_0}{v_0^2}}\right)$$

Note that as the initial velocity becomes large, or the initial height becomes small, all these formulations approach 45 degrees.

Next we seek the range attained if we use this angle. Placing Eq(2) into Eq(1), we will find a remarkably simple result (after some algebra, which this version of MCAD won't do, did it by hand, it works):

$$R_{max} = \frac{v_0}{g}\sqrt{v_0^2 + 2\,g\,y_0} = \frac{v_0\,v_f}{g} \qquad (3)$$

We would also like to know the maximum height attained during this optimal trajectory. For this we use the optimal angle in

$$y_{max} = y_0 + \frac{v_0^2}{2\,g}\sin\left(\theta_{opt}\right)^2$$

which is

$$y_{max} = y_0 + \frac{v_0^4}{4\,g\left(v_0^2 + g\,y_0\right)} \qquad (4)$$

The TOF for this trajectory is found using the optimum angle in the general TOF equation:

$$T = \frac{1}{g}\left[v_0\sin\left(\theta_{opt}\right) + \sqrt{\left(v_0\sin\left(\theta_{opt}\right)\right)^2 + 2\,g\,y_0}\right]$$

Again MCAD will not do all the algebra, and the result, by hand, is

$$T_{opt} = \frac{1}{g}\sqrt{2\left(v_0^2 + g\,y_0\right)} \qquad (5)$$

Another way to get this is to solve for T in

$$R_{max} = v_0\cos\left(\theta_{opt}\right)T_{opt}$$

$$\frac{\frac{v_0}{g}\sqrt{v_0^2 + 2\,g\,y_0}}{v_0\cos\left(\theta_{opt}\right)} = \frac{1}{g}\,\frac{\sqrt{v_0^2 + 2\,g\,y_0}}{\frac{\sqrt{v_0^2+2\,g\,y_0}}{\sqrt{2\left(v_0^2+g\,y_0\right)}}} = \frac{1}{g}\sqrt{2\left(v_0^2 + g\,y_0\right)}$$

This uses the triangle for the initial and final velocities, for the optimal angle (see the arccosine version).

We have already seen that the initial and final velocities are perpendicular, but let's try to prove it. At the end of the trajectory, when the x-coordinate is the range (by definition), the derivative of y(x) gives the slope of the velocity vector. This derivative, anywhere on the trajectory, is

$$\frac{dy}{dx} = \frac{-g}{\left(v_0 \cos(\theta)\right)^2} x + \tan(\theta)$$

So if we use the range, derived above, for x, we get

$$\frac{\left(-\sqrt{v_0^{\,2} + 2\, g\, y_0} + \sin(\theta)\cos(\theta)\, v_0\right)}{v_0 \cos(\theta)^2}$$

and then use the optimal angle in this; we then have the derivative (slope) at the endpoint:

$$\frac{-\sqrt{v_0^{\,2} + 2\, g\, y_0}}{v_0} \qquad\qquad \theta_R = \operatorname{atan}\left(\frac{-\sqrt{v_0^{\,2} + 2\, g\, y_0}}{v_0}\right) \tag{6}$$

We see from the derivative above that the initial slope is just the tangent of the optimal angle, so that, from Eq(2),

$$\frac{v_0}{\sqrt{v_0^{\,2} + 2\, g\, y_0}}$$

Now we have that these slopes are negative inversely related, which is the condition for these tangent lines to be perpendicular. It is also the case that if we add these angles

$$\operatorname{atan}\left(\frac{v_0}{\sqrt{v_0^{\,2} + 2\, g\, y_0}}\right) + \operatorname{atan}\left(\frac{\sqrt{v_0^{\,2} + 2\, g\, y_0}}{v_0}\right)$$

we get 90 degrees. Note that the minus was not used in the final angle, since this angle is interior to the triangle defined by the horizontal and the velocity vectors. If these two angles add to 90 degrees then of course the remaining angle must be 90 degrees, and the vectors are perpendicular.

To complete this analysis, we find the intersection point of the velocity vectors. This occurs when the initial vector and final vector y-values are equal.

$$y_1(x) := \tan(\theta)\, x + y_0 \qquad\qquad y_2(x) := \tan(\theta_R)\,(x - R)$$

slope intercept point (endpoint, at x=R) slope (final velocity vector)

Setting these equal, we get

$$x_{intersect} = \frac{y_0 + \tan(\theta_R)\, R_{max}}{\tan(\theta_R) - \tan(\theta_{opt})} \qquad\qquad y_{intersect} = \tan(\theta_{opt})\, x_{intersect} + y_0$$

If we substitute the definitions for these quantities, and do the algebra, we find some much simpler and interesting results, namely:

$$x_{intersect} = \frac{v_0}{2\ g} \sqrt{v_0^{\ 2} + 2\ g\ y_0} \qquad\qquad y_{intersect} = y_0 + \frac{v_0^{\ 2}}{2\ g} \tag{7}$$

The x intersection point is at one-half the maximum range, Eq(3). The y intersection happens to fall on the directrix of the trajectory (this is defined in the Parabolic Trajectory paper).