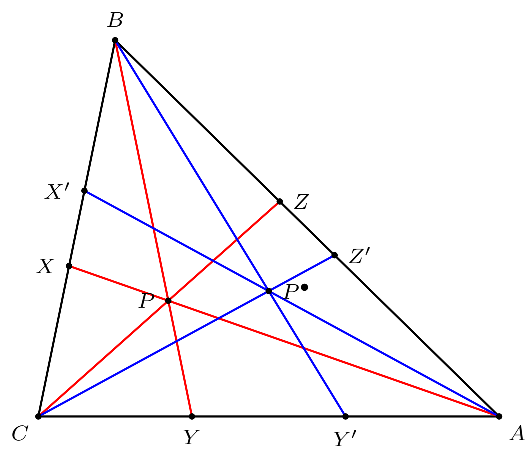
Chapter 12

Isotomic and isogonal conjugates

12.1 Isotomic conjugates

The Gergonne and Nagel points are examples of isotomic conjugates. Two points P and Q (not on any of the side lines of the reference triangle) are said to be isotomic conjugates if their respective traces are symmetric with respect to the midpoints of the corresponding sides. Thus,

$$BX = X'C, \qquad CY = Y'A, \qquad AZ = Z'B.$$

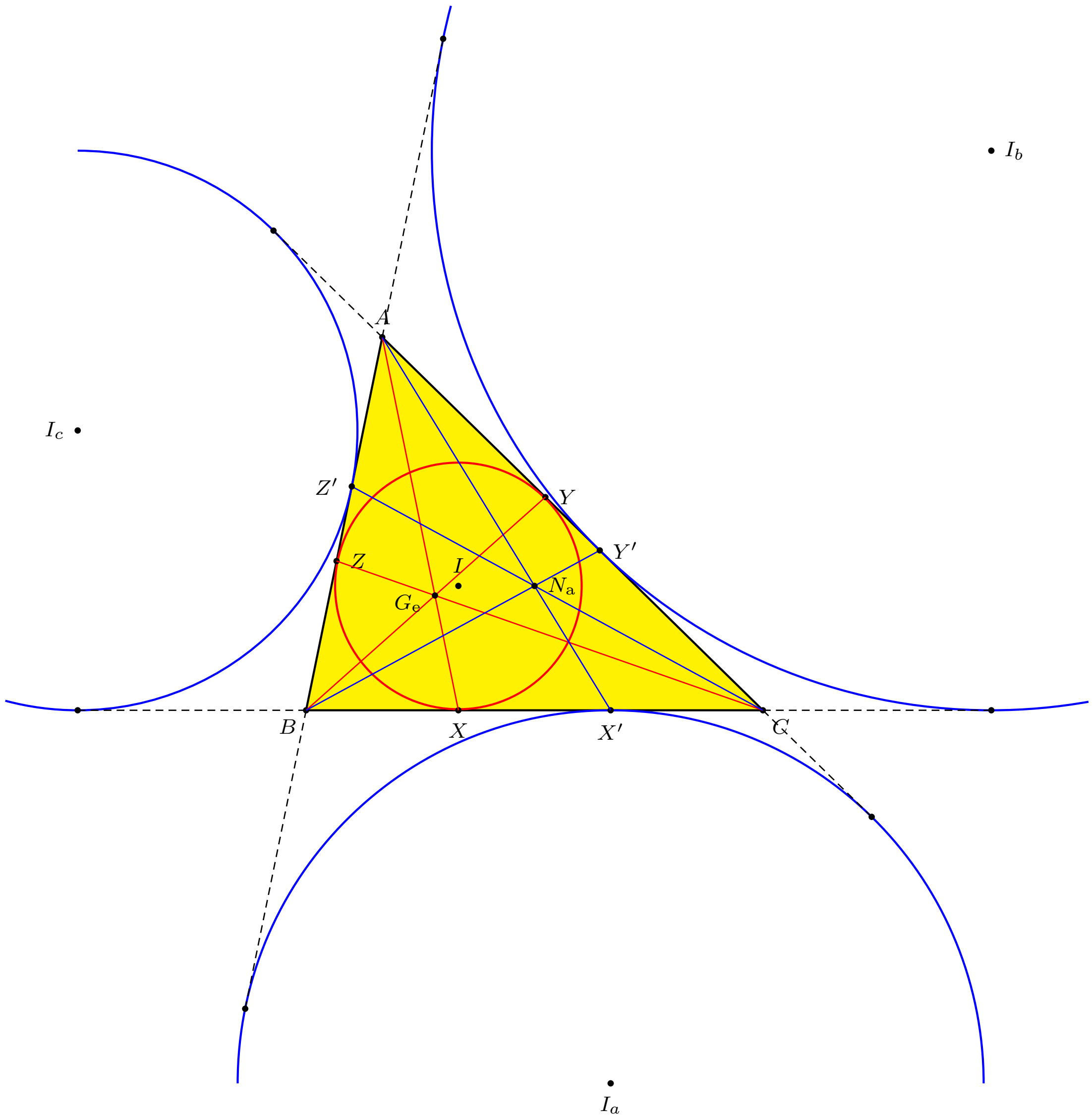


We shall denote the *isotomic conjugate* of P by $P^\bullet$. If $P = (x : y : z)$, then

$$P^\bullet = \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right) = (yz : zx : xy).$$

12.1.1 The Gergonne and Nagel points

$$G_{\mathrm{e}} = \left(\frac{1}{s-a} : \frac{1}{s-b} : \frac{1}{s-c}\right), \quad N_{\mathrm{a}} = (s-a : s- : s-c).$$

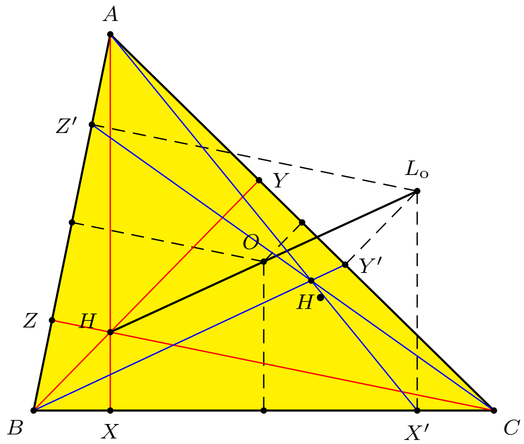


12.1.2 The isotomic conjugate of the orthocenter

The isotomic conjugate of the orthocenter is the point

$$H^\bullet = (b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2).$$

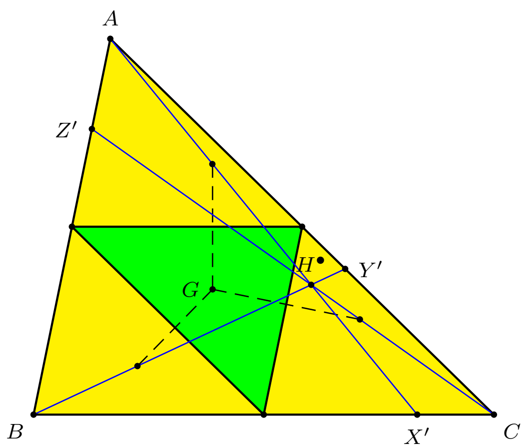
Its traces are the pedals of the deLongchamps point L_o, the reflection of H in O.



Exercise

1. Let XYZ be the cevian triangle of $H^\bullet$. Show that the lines joining X, Y, Z to the midpoints of the corresponding altitudes are concurrent. What is the common point? [1]

2. Show that $H^\bullet$ is the perspector of the triangle of reflections of the centroid G in the sidelines of the medial triangle.



[1] The centroid. Apply Menelaus theorem to triangle AA_HD, and transversal MX, where M is the midpoint of the altitude AA_H.

12.1.3 Congruent-parallelians point

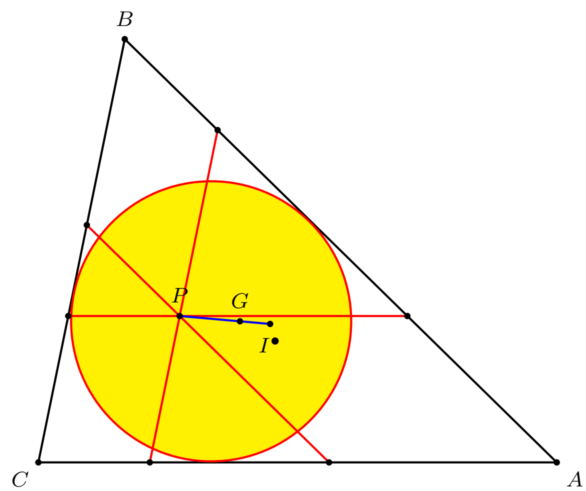
Given triangle ABC, we want to construct a point P the three lines through which parallel to the sides cut out equal intercepts. Let $P = xA + yB + zC$ in absolute barycentric coordinates. The parallel to BC cuts out an intercept of length $(1-x)a$. It follows that the three intercepts parallel to the sides are equal if and only if

$$1-x : 1-y : 1-z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

The right hand side clearly gives the homogeneous barycentric coordinates of $I^\bullet$, the isotomic conjugate of the incenter I. [2] This is a point we can easily construct. Now, translating into *absolute* barycentric coordinates:

$$I^\bullet = \frac{1}{2}[(1-x)A + (1-y)B + (1-z)C] = \frac{1}{2}(3G - P).$$

we obtain $P = 3G - 2I^\bullet$, and can be easily constructed as the point dividing the segment $I^\bullet G$ externally in the ratio $I^\bullet P : PG = 3 : -2$. The point P is called the congruent-parallelians point of triangle ABC. [3]



Exercise

1. Calculate the homogeneous barycentric coordinates of the congruent-parallelian point and the length of the equal parallelians. [4]

2. Let $A'B'C'$ be the midway triangle of a point P. The line $B'C'$ intersects CA at

$$\begin{array}{ll} B_a = B'C' \cap CA, & C_a = B'C' \cap AB, \\ C_b = C'A' \cap AB, & A_b = C'A' \cap BC, \\ A_c = A'B' \cap BC, & B_c = A'B' \cap CA. \end{array}$$

Determine P for which the three segments B_aC_a, C_bA_b and A_cB_c have equal lengths.

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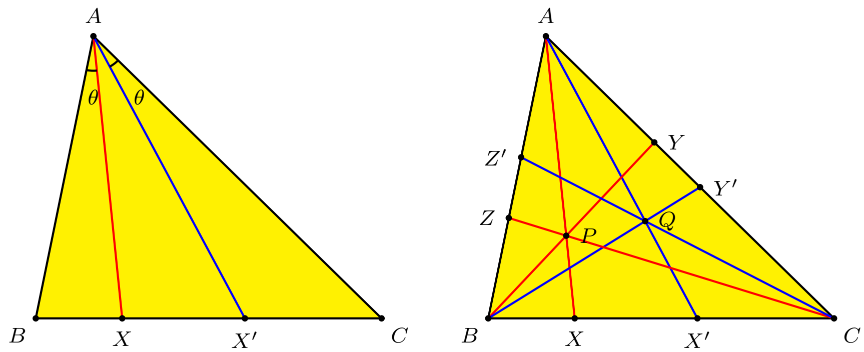
[2] The isotomic conjugate of the incenter appears in ETC as the point X_{75}.

[3] This point appears in ETC as the point X_{192}.

[4] $(ca + ab - bc : ab + bc - ca : bc + ca - ab)$. The common length of the equal parallelians is $\frac{2abc}{ab+bc+ca}$.

12.2 Isogonal conjugates

Let P be a point with homogeneous barycentric coordinates $(x : y : z)$.



Let X and X' be points on BC such that the lines AX and AX' are isogonal, ***i.e.***, $\angle BAX = \angle X'AC$. If this common angle is θ, then

$$\frac{BX}{XC} = \frac{c}{b} \cdot \frac{\sin\theta}{\sin(A-\theta)} \quad \text{and} \quad \frac{BX'}{X'C} = \frac{c}{b} \cdot \frac{\sin(A-\theta)}{\sin\theta}.$$

From this $\frac{BX}{XC} \cdot \frac{BX'}{X'C} = \frac{c^2}{b^2}$.

Since $X = (0 : y : z)$, $\frac{BX}{XC} = \frac{z}{y}$, we have $\frac{BX'}{X'C} = \frac{c^2}{b^2} \cdot \frac{z}{y} = \frac{\frac{c^2}{z}}{\frac{b^2}{y}}$. The point X' has coordinates $\left(0 : \frac{b^2}{y} : \frac{c^2}{z}\right)$.

Similarly, the reflections of the cevians BP and CP in the respective angle bisectors intersect CA at $Y' = \left(\frac{a^2}{x} : 0 : \frac{c^2}{z}\right)$ and AB at $Z' = \left(\frac{a^2}{x} : \frac{b^2}{y} : 0\right)$. The points X', Y', Z' are the traces of

$$P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right) = (a^2yz : b^2zx : c^2xy).$$

The point P^* is called the *isogonal conjugate* of P. Clearly, P is the isogonal conjugate of P^*.

Example. It is easy to see that the incenter is its own isogonal conjugate, so are the excenters.

12.3 Examples of isogonal conjugates

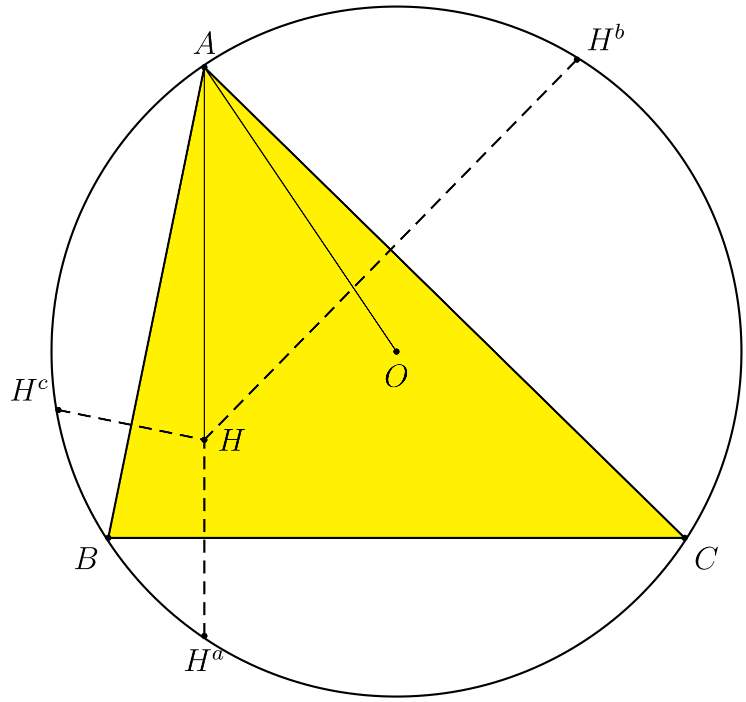
12.3.1 The circumcenter and orthocenter

The circumcenter O and the orthocenter H are isogonal conjugates, since

$$\angle(AO, AC) = \frac{\pi}{2} - \beta = -\angle(AH, AB),$$

and similarly $\angle(BO, BA) = -\angle(BH, BC)$, $\angle(CO, CB) = -\angle(CH, CA)$.

Since $OO^a = AH$, AOO^aH is a parallelogram, and $HO^a = AO$. This means that the circle of reflections of O is congruent to the circumcircle. Therefore, the circle of reflections of H is the circumcircle, and the reflections of H lie on the circumcircle.



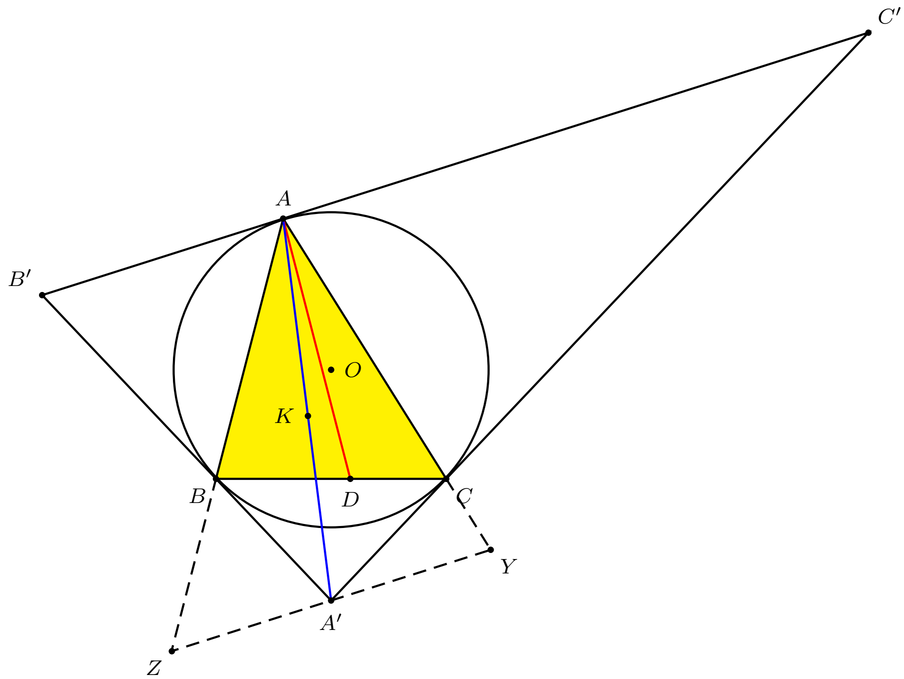
12.3.2 The isogonal conjugates of the Gergonne and Nagel points

Proposition 12.1. (a) *The isogonal conjugate of the Gergonne point is the insimilicenter of the circumcircle and the incircle:* $G_e^* = T_+$.
(b) *The isogonal conjugate of the Nagel point is the exsimilicenter of the circumcircle and the incircle:* $N_a^* = T_-$.

12.4 The symmedian point

The isogonal conjugate of the centroid $G = (1 : 1 : 1)$ is called the *symmedian point*, usually denoted by K. It is the point of concurrency of the symmedians, which are the isogonal lines of the medians. It has homogeneous barycentric coordinates $(a^2 : b^2 : c^2)$.

Theorem 12.1. *The symmedian point is the perspector of the tangential triangle.*



Proof. Let $A'B'C'$ be the tangential triangle, so that $B'C'$, $C'A'$, $A'B'$ are the tangents to the circumcircle at the vertices A, B, C respectively. Extend AB and AC to Z and Y such that $A'B = A'Z$ and $A'C = A'Y$. We claim that Y, A', Z are collinear. Note that

$$\begin{aligned}
\angle ZA'B &= 180^\circ - 2\angle A'BZ = 180^\circ - 2\angle B'BA = 180^\circ - 2C,\\
\angle BA'C &= 180^\circ - 2\angle A'BC = 180^\circ - 2A,\\
\angle CA'Y &= 180^\circ - 2\angle A'CY = 180^\circ - 2\angle ACC' = 180^\circ - 2B.
\end{aligned}$$

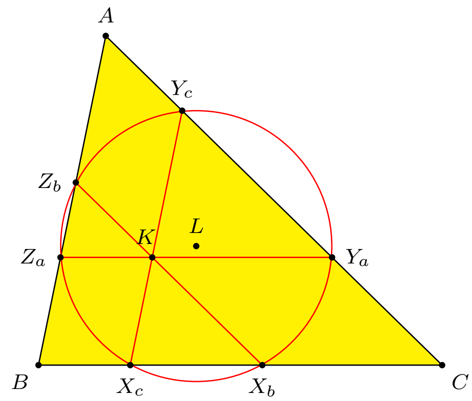
Hence,

$$\angle ZA'B + \angle BA'C + \angle CA'Y = 540^\circ - 2(A + B + C) = 180^\circ,$$

and Y, A', Z are collinear. It follows that
(i) AA' is a median of triangle AYZ, and
(ii) AYZ and ABC are similar.
If D is the midpoint of BC, then the triangles $AA'Z$ and ADC are similar. Therefore, AA' and AD are isogonal lines with respect to AB and AC. Similarly, the BB' and CC' are isogonal to B- and C-medians. The lines AA', BB', CC' therefore intersect at the isogonal conjugate of the centroid G, which is the point K. □

12.4.1 The first Lemoine circle

Given triangle ABC, how can one choose a point P so that when parallel lines are constructed through it to intersect each sideline at two points, the resulting six points are on a circle?



Suppose $P = (u : v : w)$ in homogeneous barycentric coordinates.
(1) $AY_a : AC = AZ_a : AB = v + w : u + v + w$, so that
$AY_a = \frac{b(v+w)}{u+v+w}$ and $AZ_a = \frac{c(v+w)}{u+v+w)}$;
(2) $AY_c : AC = w : u + v + w$, so that $AY_c = \frac{bw}{u+v+w}$,
(3) $AZ_b : AB = v : u + v + w$, so that $AZ_b = \frac{cv}{u+v+w}$.
Since Y_c, Y_a, Z_a, Z_b are concyclic, by the intersection chords theorem,

$$\begin{aligned} & AY_a \cdot AY_c = AZ_a \cdot AZ_b \\ \Longrightarrow \; & \frac{b(v+w)}{u+v+w} \cdot \frac{bw}{u+v+w} = \frac{c(v+w)}{u+v+w} \cdot \frac{cv}{u+v+w}. \end{aligned}$$

Therefore, $v : w = b^2 : c^2$. Similarly, since X_b, X_c, Y_c, Y_a are concyclic, $u : v = a^2 : b^2$. From this we conclude if the six points are concyclic, P must be the symmedian point $K = (a^2 : b^2 : c^2)$.

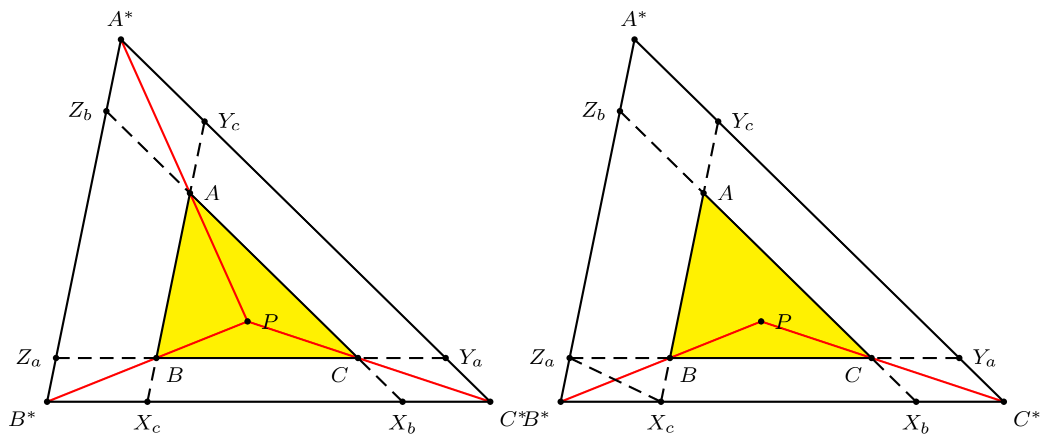
Conversely, if P is the symmedian point, then there are circles
(i) $\mathcal{C}_a$ passing through Y_c, Y_a, Z_a, Z_b,
(ii) $\mathcal{C}_b$ passing through Z_a, Z_b, X_b, X_c,
(iii) $\mathcal{C}_c$ passing through X_b, X_c, Y_c, Y_a.
This is an impossibility, since the radical axes of three circles are either parallel or concurrent. Therefore, the six points are concyclic, and the circle containing them is called the *first Lemoine circle*.

12.5 Triangles bounded by lines parallel to the sidelines

Theorem 12.2 (Homothetic center theorem). *If parallel lines X_bX_c, Y_cY_a, Z_aZ_b to the sides BC, CA, AB of triangle ABC are constructed such that*

$$AB : BX_c = AC : CX_b = 1 : t_1,$$
$$BC : CY_a = BA : AY_c = 1 : t_2,$$
$$CA : AZ_b = CB : BZ_a = 1 : t_3,$$

these lines bound a triangle $A^*B^*C^*$ homothetic to ABC with homothety ratio $1 + t_1 + t_2 + t_3$. The homothetic center is a point P with homogeneous barycentric coordinates $t_1 : t_2 : t_3$.



Proof. Let P be the intersection of B^*B and C^*C. Since

$$\begin{aligned} B^*C^* &= B^*X_c + X_cX_b + X_bC^* \\ &= t_3a + (1 + t_1)a + t_2a = (1 + t_1 + t_2 + t_3)a, \end{aligned}$$

we we have

$$PB : PB^* = PC : PC^* = 1 : 1 + t_1 + t_2 + t_3.$$

A similar calculation shows that AA^* and BB^* intersect at the same point P. This shows that $A^*B^*C^*$ is the image of ABC under the homothety $\mathsf{h}(P,\ 1 + t_1 + t_2 + t_3)$.

Now we compare areas. Note that

(1) $\Delta(BZ_aX_c) = \frac{BX_c}{AB} \cdot \frac{BZ_a}{CB} \cdot \Delta(ABC) = t_1t_3\Delta(ABC)$,

(2) $\frac{\Delta(PBC)}{\Delta(BZ_aB^*)} = \frac{PB}{BB^*} \cdot \frac{CB}{BZ_a} = \frac{1}{t_1+t_2+t_3} \cdot \frac{1}{t_3} = \frac{1}{t_3(t_1+t_2+t_3)}$.

Since $\Delta(BZ_aB^*) = \Delta(BZ_aX_c)$, we have $\Delta(PBC) = \frac{t_1}{t_1+t_2+t_3} \cdot \Delta(ABC)$.

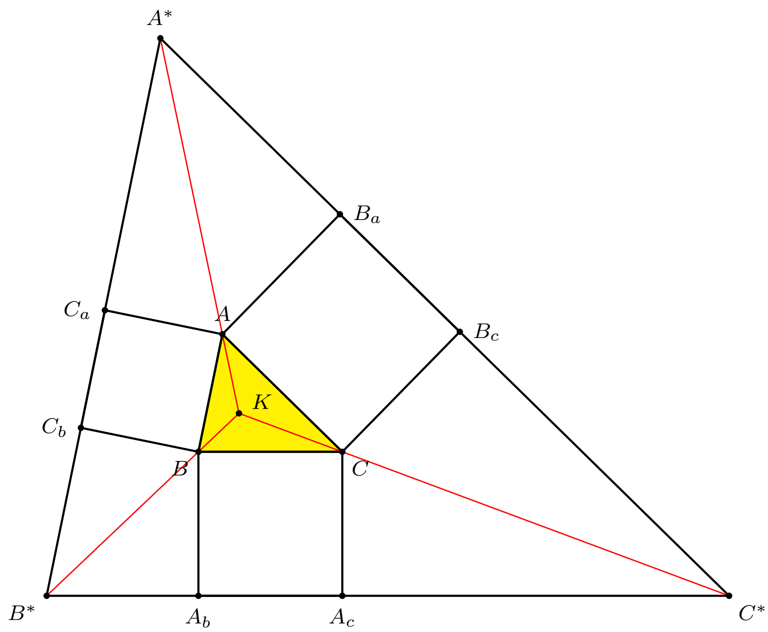
Similarly, $\Delta(PCA) = \frac{t_2}{t_1+t_2+t_3} \cdot \Delta(ABC)$ and $\Delta(PAB) = \frac{t_3}{t_1+t_2+t_3} \cdot \Delta(ABC)$. It follows that

$$\Delta(PBC) : \Delta(PCA) : \Delta(PAB) = t_1 : t_2 : t_3.$$

□

12.5.1 The symmedian point

Consider the square erected externally on the side BC of triangle ABC, The line containing the outer edge of the square is the image of BC under the homothety $\mathsf{h}(A, 1+t_1)$, where $1+t_1 = \frac{\frac{2\Delta}{a}+a}{\frac{2\Delta}{a}} = 1+\frac{a^2}{2\Delta}$, *i.e.*, $t_1 = \frac{a^2}{2\Delta}$. Similarly, if we erect squares externally on the other two sides, the outer edges of these squares are on the lines which are the images of CA, AB under the homotheties $\mathsf{h}(B, 1+t_2)$ and $\mathsf{h}(C, 1+t_3)$ with $t_2 = \frac{b^2}{2\Delta}$ and $t_3 = \frac{c^2}{2\Delta}$.



The triangle bounded by the lines containing these outer edges is called the *Grebe triangle* of ABC. It is homothetic to ABC at

$$\left(\frac{a^2}{2\Delta} : \frac{b^2}{2\Delta} : \frac{c^2}{2\Delta}\right) = (a^2 : b^2 : c^2),$$

the symmedian point K, and the ratio of homothety is

$$1 + (t_1 + t_2 + t_3) = \frac{2\Delta + a^2 + b^2 + c^2}{2\Delta}.$$

Remark. Note that the homothetic center remains unchanged if we replaced t_1, t_2, t_3 by kt_1, kt_2, kt_3 for the same nonzero k. This means that if similar rectangles are constructed on the sides of triangle ABC, the lines containing their outer edges always bound a triangle with homothetic center K.