Class 2 §5.5

u-substitution

IDEA: Take an integral with a "complicated" expression (e.g. something is "trapped") and pretend it's simple.

(but this has consequences!)

Ex: $\int x^3 \cos(\underbrace{x^4+2}_{\text{"trapped"}})\,dx$

"Pretend" $x^4+2 = u(x)$ ①

$\Rightarrow \int x^3 \cos(u)\,dx$

Trick: Compute $\frac{du}{dx} = 4x^3$

Pretend $\frac{du}{dx}$ is a fraction to get

$$du = 4x^3\,dx$$

$$\Rightarrow \frac{1}{4}\,du = x^3\,dx \quad \text{II}$$

I + II

$$\int x^3 \cos(x^4+2)\,dx = \int \cos(x^4+2)\,x^3\,dx$$

$$= \int \cos(u)\cdot\frac{1}{4}\,du = \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{1}{4}\sin(u) + C$$

$$= \frac{1}{4}\sin(x^4+2) + C$$

Check: $\frac{d}{dx}\left(\frac{1}{4}\sin(x^4+2)+C\right)$

$$= \frac{1}{4}\frac{d}{dx}\left(\sin(x^4+2)\right)$$

$$= \frac{1}{\cancel{4}} \cos(x^4+2) \cdot \cancel{4} x^3$$

$$= \cos(x^4+2)\, x^3$$

Ex: $\int \sqrt{2x+1}\, dx$

"trapped"

Pretend:

$$u(x) = 2x+1$$

$$\Rightarrow \frac{du}{dx} = 2$$

$$\Rightarrow \frac{1}{2} du = dx$$

$$\Rightarrow \int \sqrt{u} \cdot \frac{1}{2} du = \frac{1}{2} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

<u>Check:</u> $\frac{d}{dx}\left(\frac{1}{3}(2x+1)^{3/2} + C\right)$

chain rule

$$= \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}}(2x+1)^{\frac{1}{2}} \cdot \cancel{2}$$

$$= \sqrt{2x+1}$$

<u>Chain rule in reverse</u>

$$(f \circ g)' = \underbrace{f'(g)}_{\frac{df}{dg}} \cdot \underbrace{g'}_{\frac{dg}{dx}} = \frac{df}{dg}\frac{dg}{dx}$$

$f(g)$

Want to integrate

$$\int f'(g(x)) \cdot g'(x)\, dx$$

$$= \int (f \circ g)'\, dx$$

BY THE CHAIN RULE!

$$= (f \circ g) + C$$

BY FTC

<u>Ex:</u> Compute $\int \tan(x)\, dx$

$$= \int \frac{\sin(x)}{\cos(x)}\, dx$$

du

u

$u = \cos(x)$

$du = -\sin(x)\, dx$

$$= \int \frac{-1}{u}\, du$$

$$= -\ln|u| + C = \ln|u^{-1}| + C$$

$$\boxed{\begin{gathered} \frac{1}{\cos(x)} = (\cos(x))^{-1} \\ \neq \cos^{-1}(x) \\ = \arccos(x) \\ \text{CAREFUL!} \end{gathered}}$$

$$= \ln\left|\frac{1}{u}\right| + C$$

$$= \ln\left|\frac{1}{\cos(x)}\right| + C$$

$$= \ln|\sec(x)| + C$$

Ex: $$\int_0^1 e^{5x}\,dx$$

$u = 5x$

$\dfrac{du}{dx} = 5$

$dx = \frac{1}{5}\,du$

$$\overset{?}{=} \cancel{\int_0^1 e^u\,du}$$

Compute the antiderivative w/o bounds

$$\int e^{5x}\,dx = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u = \frac{1}{5}e^{5x}$$

$$\Longrightarrow \int_0^1 e^{5x}\,dx = \frac{1}{5}e^{5x}\Big|_0^1 = \frac{1}{5}e^5 - \frac{1}{5}$$

General Shortcut

$$\int_a^b f'(\underbrace{g(x)}_{u(x)}) \cdot \underbrace{g'(x)}_{\frac{du}{dx}}\,dx = \int_a^b f'(u)\frac{du}{dx}\,dx$$

$$= (f \circ g)(x)\Big|_a^b = f(g(b)) - f(g(a))$$

$$= f(u)\Big|_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f'(u)\,du$$

Ex: $\displaystyle\int^e \frac{\ln(x)}{x}\,dx$ $\qquad u = \ln(x)$

$\frac{du}{} = \frac{1}{}$

$$1 \qquad \qquad \frac{\quad}{dx} = \frac{\quad}{x}$$

$$du = \frac{dx}{x}$$

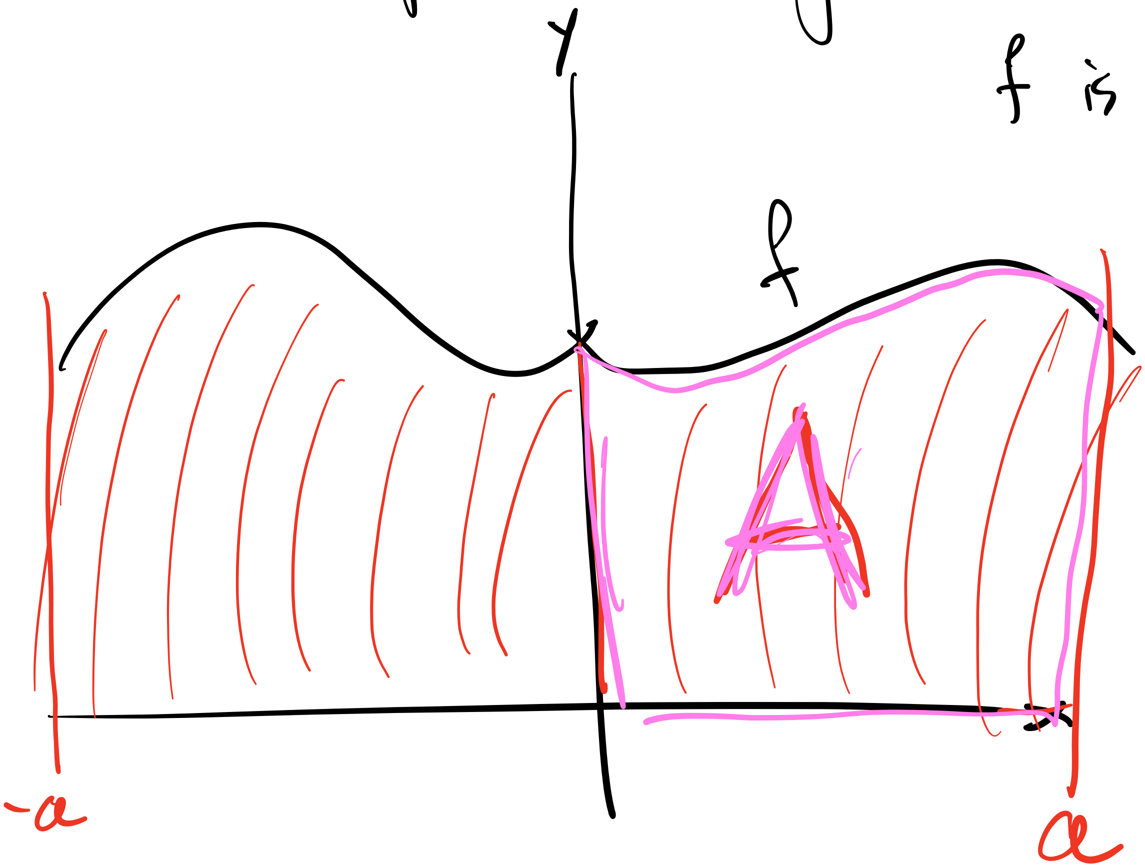
$$= \int_{u(1)}^{u(e)} u \, du$$

$$= \int_0^1 u \, du = \frac{1}{2} u^2 \Big|_0^1 = \frac{1}{2}$$

TAKEAWAY: When doing u-sub on an integral w/ bounds, after replacing the bounds correctly, we don't need to "back substitute".

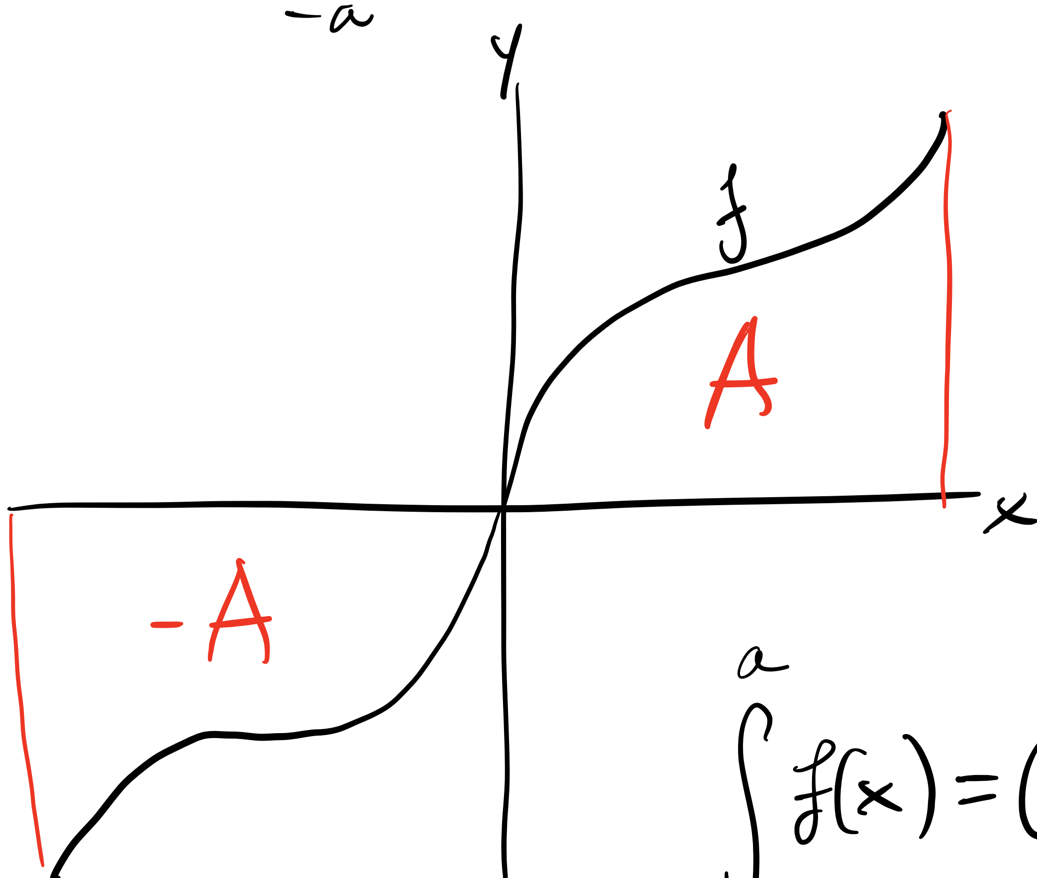
Application: Using symmetry to compute integrals

f is even



$$\int_{-a}^{a} f(x)\,dx = 2A$$

f odd



$$\int_{-a}^{a} f(x) = 0$$

- f is a function

$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}_{\mathrm{I}} + \int_{0}^{a} f(x)\,dx$$

Use u-sub on I

($u = -x$ so $du = -dx$
$x = -u$ or $dx = -du$)

$$\mathrm{I} = -\int_{u(-a)}^{u(0)} f(-u)\,du$$ (subbing in)

$$= -\int_{a}^{0} f(-u)\,du = \int_{0}^{a} f(-u)\,du$$

$$= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

$0 \qquad 0$

Case i: Say f is even, so $f(-x)=f(x)$

$$\Longrightarrow \int_0^a f(u)\,du + \int_0^a f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx$$

Case ii: f odd, so $f(-x)=-f(x)$

$$\Longrightarrow \int_0^a -f(u)\,du + \int_0^a f(x)\,dx$$

$$= \int_0^a -f(x)+f(x)\,dx = \int_0^a 0\,dx$$

$$= 0$$