(31)	$-\theta_3$	$-\theta_2$	$-\theta_1$
(12)	$-\theta_2$	$-\theta_1$	$-\theta_3$
(123)	θ_2	θ_3	θ_1
(321)	θ_3	θ_1	θ_2
$\overline{123}$	$-\theta_2$	$-\theta_3$	$-\theta_1$
$\overline{321}$	$-\theta_3$	$-\theta_1$	$-\theta_2$

§ 42. **Geometric Solution of the Cubic Equation** – Let the given cubic equation be

$$a_0x^3+3a_1x^2+3a_2x+a_3=0$$

The problem is, given the numbers a_i, to construct the points.

There are two Hessian points h_i, and the cubic in terms of these is

$$(x-h_1)^3=\lambda(x-h_2)^3$$

Thus we take

$$1-\lambda=a_0$$
$$h_1-\lambda h_2=-a_1$$
$$h_1{}^2-\lambda h_2{}^2=a_2$$
$$h_1{}^3-\lambda h_2{}^3=-a_3$$

whence

$$a_0a_2-a_1{}^2=-\lambda(h_1-h_2)^2$$
$$a_0a_3-a_1a_2=\lambda(h_1-h_2)^2(h_1+h_2)$$
$$a_1a_3-a_2{}^2=-\lambda(h_1-h_2)^2(h_1+h_2)$$

Thus the Hessian points are given by

$$(a_0a_2-a_1{}^2)h^2+(a_0a_3-a_1a_2)h+a_1a_3-a_2{}^2=0$$

This might be written directly if we assumed the homographic theory of the cubic.

We suppose this quadratic to be solved. We have also

$$3(h_1-\lambda h_2)=(1-\lambda)(x_1+x_2+x_3)$$

Thus if g be the mean point or centroid

$$g(1-\lambda)=h_1-\lambda h_2$$

or

$$\lambda=(g-h_1)/(g-h_2)$$

The solutions are then

$$(x-h_1)/(x-h_2)=\text{a cube root of } \lambda$$

Thus

$$|\,(x-h_1)/(x-h_2)\,|=|\,\lambda^{1/3}\,|$$

This gives the circumcircle of the three points x_i; and the angle h_1xh_2 is a third of the angle h_1gh_2. This gives three

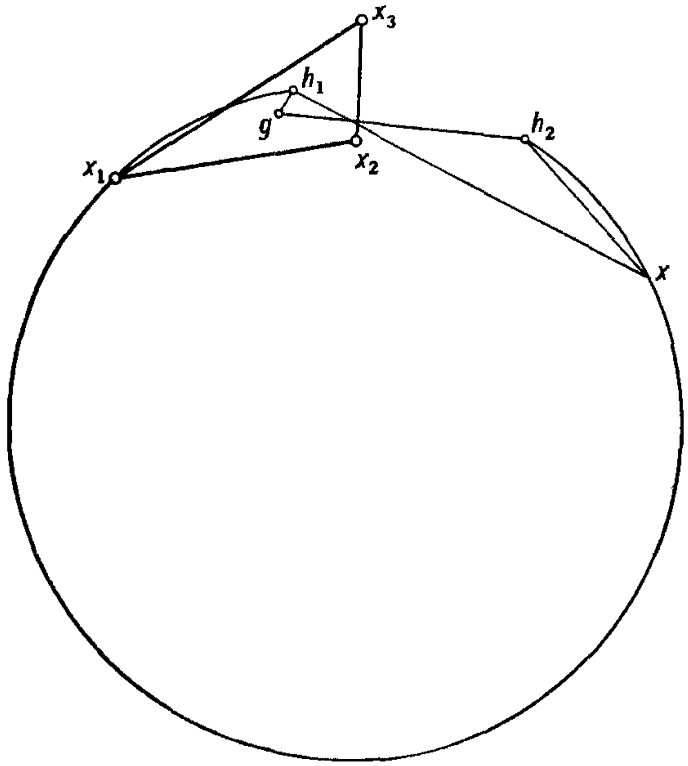

FIG. 28

arcs from h_1 to h_2, which cut the circumcircle at the points x_i (fig. 28).

§ 43. The Groups of the Rectangle and Rhombus – In § 26 we had the four inversions:

$$x = \bar{y}$$
$$x = -\bar{y}$$
$$x\bar{y} = 1$$
$$x\bar{y} = -1$$

These form a finite group of eight operations G_8. For if we combine them we get only

$$x = y$$
$$x = -y$$
$$xy = 1$$
$$xy = -1$$

Fig. 29 shows the eight fundamental regions sent into one another by the operations, one region being marked I.