The Euler line, continued

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Abstract: Previously a new notable point of plane triangles was discovered that fits into the Euler line and it has application in microwave measurements. Now this 2-dimensional result is generalized. This paper is intended to stimulate interaction between geometry and physics.

Keywords: Plane triangles, Euler line, tetrahedron, minimum sensitivity calibration

1. Introduction

Sensitivity minimization in microwave measurements led us to a notable point of plane triangles [4]. Then we analyzed the application possibility and obtained the so-called minimum sensitivity calibration [5]. In this paper we generalize the notable point for more than 2 dimensions.

In Section 2 we define the new notable point in 3 dimensions. We prove that this notable point does not fit into the Euler line [1] of the tetrahedron in general. A condition for the length of two opposite edges of the tetrahedron has been provided for the notable point to fit into the Euler line. In Section 3 the number of new notable points has been investigated. In Section 4, further generalization possibilities have been provided.

We are not aware yet of applications. According to the topics of the author, applications are expected in circuit theory or quantum communications.

Finally, we repeat here our starting point to be generalized [4], Fig. 1.1.

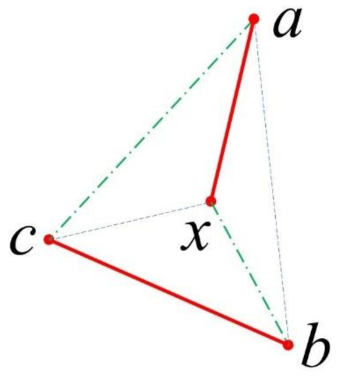

Fig. 1.1. The triangle abc, and our notable point x inside. Line sections that are necessary for definition of x, are denoted by corresponding colors and line styles

The notable point x for the triangle abc is defined by xa/bc=xb/ca=xc/ab, where we denote the line section between points a and b by ab. The number of such points x is 2,1,1 and 0 for general acute, regular, right and obtuse triangles, respectively, and the notable point fits into the Euler line.

2. Extension of our notable point for tetrahedrons

In this Section we generalize our notable point and we prove that under some conditions, it fits into the Euler line of the tetrahedron.

Our original form for triangles was

$$\frac{|\underline{x}-\underline{a}|}{|\underline{b}-\underline{c}|}=\frac{|\underline{x}-\underline{b}|}{|\underline{c}-\underline{a}|}=\frac{|\underline{x}-\underline{c}|}{|\underline{a}-\underline{b}|} \tag{2.1}$$

where $\underline{x}$ is the position vector pointing from the origin to the new notable point. Taking square of (2.1), squared section length between x and a vertex, is divided by squared length of the edge of the triangle opposite to the vertex.

For tetrahedrons, denominator is modified to the area of the face opposite to the vertex (Fig. 4.1):

$$\frac{|\underline{x}-\underline{a}|^2}{A_{bcd}^2}=\frac{|\underline{x}-\underline{b}|^2}{A_{cda}^2}=\frac{|\underline{x}-\underline{c}|^2}{A_{dab}^2}=\frac{|\underline{x}-\underline{d}|^2}{A_{abc}^2} \tag{2.2}$$

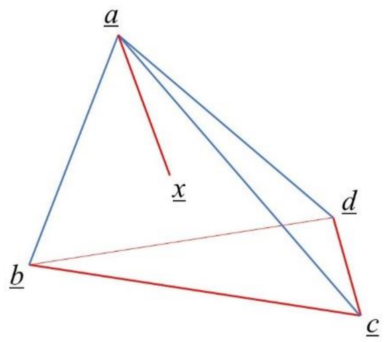

Fig. 2.1. A tetrahedron, with a point x inside. The section and face in (2.2), leftmost expression, are colored in red

Definition 1: Given the tetrahedron abcd, a notable point $\underline{x}$ is defined by (2.2).

Theorem 1: The notable point $\underline{x}$ fits into the Euler line, if it exists, and the tetrahedron has at least one pair of opposite edges that satisfy some conditions (2.35), (2.37-38).

Proof:

Area of the triangle abc, denoted as A_{abc}, can be expressed as

$$A_{abc}=\left|\underline{a}\ x\ \underline{b}+\underline{b}\ x\ \underline{c}+\underline{c}\ x\ \underline{a}\right| \tag{2.3}$$

where $\underline{a}\ x\ \underline{b}$ denotes the cross product of $\underline{a}, \underline{b}$. With (2.2-3),

$$\left(\underline{x}-\underline{a}\right)^2=\mu(\underline{b}\ x\ \underline{c}+\underline{c}\ x\ \underline{d}+\underline{d}\ x\ \underline{b})^2 \tag{2.4}$$

$$\left(\underline{x}-\underline{b}\right)^2=\mu(\underline{c}\ x\ \underline{d}+\underline{d}\ x\ \underline{a}+\underline{a}\ x\ \underline{c})^2 \tag{2.5}$$

and two more equations that we do not use now. Notation μ is a non-negative real quantity, its dimension is 1/length^2.

Take the difference of (2.4-5):

$$\left(\underline{b}-\underline{a}\right)\left(2\underline{x}-\underline{a}-\underline{b}\right)=$$

$$= \mu(\underline{b} - \underline{a})[(\underline{c} - \underline{d})\, x\, (\underline{b}\, x\, \underline{c} + \underline{c}\, x\, \underline{d} + \underline{d}\, x\, \underline{b} + \underline{c}\, x\, \underline{d} + \underline{d}\, x\, \underline{a} + \underline{a}\, x\, \underline{c})] \quad (2.6)$$

Because $\underline{b} - \underline{a} \neq \underline{0}$,

$$2\underline{x} - \underline{a} - \underline{b} = \mu(\underline{c} - \underline{d})\, x\, (\underline{b}\, x\, \underline{c} + \underline{c}\, x\, \underline{d} + \underline{d}\, x\, \underline{b} + \underline{c}\, x\, \underline{d} + \underline{d}\, x\, \underline{a} + \underline{a}\, x\, \underline{c}) + \lambda_1 \underline{v_1} \quad (2.7)$$

$$(\underline{b} - \underline{a})\underline{v_1} = 0 \quad (2.8)$$

In our paper [4], the origin was set to the altitude center (orthocenter). But the orthocenter does not always exist in case of a tetrahedron. Thus, we place the origin into the Monge point [2] that always exists [3]. The Monge point is defined as intersection of planes, that pass the midpoint of an edge and perpendicular to the opposite edge. A line crossing the Monge point can be described as

$$\underline{x} = \underline{x_0} + \nu \underline{z} \quad (2.9)$$

where $\underline{x_0}$ is the vector from the origin to the Monge point, $\underline{z}$ is a vector parallel to this line and ν is an arbitrary real number. The following relations are hold:

$$\frac{\underline{a+b}}{2} = \underline{x_0} + \nu \underline{z} \quad (2.10a)$$

$$\underline{z}(\underline{c} - \underline{d}) = 0 \quad (2.10b)$$

and vertices are permutable. That is:

$$[\underline{a} + \underline{b} - 2\underline{x_0}](\underline{c} - \underline{d}) = 0 \quad (2.11)$$

We want to put the Monge point into the origin, $\underline{x_0} = \underline{0}$:

$$(\underline{a} + \underline{b})(\underline{c} - \underline{d}) = 0 \quad (2.12)$$

Vertices are permutable:

$$(\underline{c} + \underline{d})(\underline{a} - \underline{b}) = 0 \quad (2.13)$$

From (2.12-13),

$$\underline{ac} = \underline{bd} \quad (2.14)$$

Again, vertices are permutable:

$$\underline{ab} = \underline{cd} \quad (2.15)$$

$$\underline{ad} = \underline{bc} \quad (2.16)$$

Between the 6 quantities in (2.14-16), only 3 equations are satisfied, (2.14-16), but $\underline{ab} \neq \underline{ac}, \underline{ab} \neq \underline{ad}, \underline{ac} \neq \underline{ad}$ in general.

Then, we introduce three notations:

$$k_1 = \underline{ab} = \underline{cd} \quad (2.17)$$

$$k_2 = \underline{ac} = \underline{bd} \quad (2.18)$$

$$k_3 = \underline{ad} = \underline{bc} \tag{2.19}$$

Comparing (2.8) and (2.13), a possibility is

$$\underline{v} = \underline{c} + \underline{d} \tag{2.20}$$

Now we simplify (2.7)

$$(\underline{c} - \underline{d})\ x\ (\underline{b}\ x\ \underline{c} + \underline{c}\ x\ \underline{d} + \underline{d}\ x\ \underline{b} + \underline{c}\ x\ \underline{d} + \underline{d}\ x\ \underline{a} + \underline{a}\ x\ \underline{c}) =$$

$$= (\underline{c} - \underline{d})\ x\ \left(2\underline{c}\ x\ \underline{d} + (-\underline{c} + \underline{d})x\ \underline{b} + \underline{(d - c)}x\ \underline{a}\right) =$$

$$= (\underline{c} - \underline{d})\ x \left(2\underline{c}\ x\ \underline{d} + (\underline{d} - \underline{c})x\ (\underline{b} + \underline{a})\right) \tag{2.21}$$

A known identity:

$$\underline{p}\ x\ (\underline{r}\ x\ \underline{s}) = \underline{r}\left(\underline{p}\,.\,\underline{s}\right) - \underline{s}(\underline{p}.\underline{r}) \tag{2.22}$$

where the dot notation means a dot (scalar) product.

(2.22) is applied for (2.21):

$$(\underline{c} - \underline{d})\ x((\underline{d} - \underline{c})x\ (\underline{b} + \underline{a})) = (\underline{d} - \underline{c})\left((\underline{c} - \underline{d})(\underline{b} + \underline{a})\right) - (\underline{b} + \underline{a})\left((\underline{c} - \underline{d})(\underline{d} - \underline{c})\right) = (\underline{b} + \underline{a})(\underline{d} - \underline{c})^2 \tag{2.23}$$

because of (2.12). Then (2.7), (2.20) and (2.23) yield

$$2\underline{x} - \underline{a} - \underline{b} = \mu\left[(\underline{c} - \underline{d})\ x\ (2\underline{c}\ x\ \underline{d}) + (\underline{b} + \underline{a})(\underline{d} - \underline{c})^2\right] + \lambda_1(\underline{c} + \underline{d}) \tag{2.24}$$

Vertices are permuted:

$$2\underline{x} - \underline{c} - \underline{d} = \mu\left[(\underline{a} - \underline{b})\ x\ (2\underline{a}\ x\ \underline{b}) + (\underline{d} + \underline{c})(\underline{b} - \underline{a})^2\right] + \lambda_2(\underline{a} + \underline{b}) \tag{2.25}$$

Difference of the last two equations is the following:

$$\underline{0} = (\underline{a} + \underline{b})\left(\mu(\underline{d} - \underline{c})^2 + 1\right) - (\underline{c} + \underline{d})\left(\mu(\underline{b} - \underline{a})^2 + 1\right) + \lambda_1(\underline{c} + \underline{d}) - \lambda_2(\underline{a} + \underline{b}) + \mu[(\underline{c} - \underline{d})\ x\ (2\underline{c}\ x\ \underline{d}) - (\underline{a} - \underline{b})\ x\ (2\underline{a}\ x\ \underline{b})] \tag{2.26}$$

$$(\underline{c} - \underline{d})\ x\ (2\underline{c}\ x\ \underline{d}) = \alpha_1(\underline{c} + \underline{d}) \tag{2.27}$$

$$(\underline{a} - \underline{b})\ x\ (2\underline{a}\ x\ \underline{b}) = \alpha_2(\underline{a} + \underline{b}) \tag{2.28}$$

We return to (2.27-28) in Appendix A2.

As $\underline{a} + \underline{b}$ and $\underline{c} + \underline{d}$ are linearly independent:

$$\mu(\underline{a} - \underline{b})^2 + 1 = \lambda_1 + \mu\alpha_1 \tag{2.29}$$

$$\mu(\underline{c} - \underline{d})^2 + 1 = \lambda_2 + \mu\alpha_2 \tag{2.30}$$

Substitution into (2.24) will result in

$$2\underline{x} = (\underline{a} + \underline{b} + \underline{c} + \underline{d}) + \mu\left[(\underline{b} + \underline{a})(\underline{d} - \underline{c})^2 + (\underline{c} + \underline{d})(\underline{a} - \underline{b})^2\right] \tag{2.31}$$

For $\mu = 0$, (2.31) gives the center of the circumscribed sphere [3]. Let us see when it crosses the Monge point:

$$\underline{0} = (\underline{a} + \underline{b} + \underline{c} + \underline{d}) + \mu\left[(\underline{b} + \underline{a})(\underline{d} - \underline{c})^2 + (\underline{c} + \underline{d})(\underline{a} - \underline{b})^2\right] \tag{2.32}$$

The necessary and sufficient condition is that the bracketed vector should be parallel to $\underline{a} + \underline{b} + \underline{c} + \underline{d}$.

$\underline{a} + \underline{b}, \underline{c} + \underline{d}$ are linearly independent:

$$0 = 1 + \mu(\underline{d} - \underline{c})^2 \tag{2.33}$$

$$0 = 1 + \mu(\underline{a} - \underline{b})^2 \tag{2.34}$$

From (2.33-34):

$$(\underline{d} - \underline{c})^2 = (\underline{a} - \underline{b})^2 \tag{2.35}$$

With all these,

$$2\underline{x} = (\underline{a} + \underline{b} + \underline{c} + \underline{d}) + \mu(\underline{a} + \underline{b} + \underline{c} + \underline{d})(\underline{d} - \underline{c})^2 \tag{2.36}$$

Now we return to (2.27-28). From (2.27) and (2.22) follows that (A2. Appendix)

$$|\underline{c}|^2 = |\underline{d}|^2 \tag{2.37}$$

Similarly, from (2.28):

$$|\underline{a}|^2 = |\underline{b}|^2 \tag{2.38}$$

There are opposite edges that should have a symmetric position with respect to the Monge point.

Although from (2.2), $\mu \geq 0$ follows, we extend the line to negative μ as well.

As we investigate Euler line, we list our notations for notable points in Table 1:

Name of the notable point	actual value for vector $\underline{x}$	actual value for parameter μ
Monge center $\boldsymbol{\mathcal{M}}$	$\underline{x_0}$	μ_0
weight center (centroid) $\boldsymbol{\mathcal{G}}$	$\underline{x_1}$	μ_1
center of circumscribed sphere (circumcenter) $\boldsymbol{\mathcal{C}}$	$\underline{x_2}$	μ_2
intersection of the Euler line and a face of the tetrahedron	$\underline{x_3}$	μ_3

The line in (2.37) crosses the origin for μ_0:

$$2\underline{x_0} = \underline{0} = (\underline{a} + \underline{b} + \underline{c} + \underline{d}) + \mu_0(\underline{a} + \underline{b} + \underline{c} + \underline{d})(\underline{d} - \underline{c})^2 \tag{2.39}$$

$$\mu_0 = -\frac{1}{(\underline{d} - \underline{c})^2} \tag{2.40}$$

The weight center is $\underline{x_1}$, and it corresponds to μ_1:

$$\underline{x_1} = \frac{\underline{a}+\underline{b}+\underline{c}+\underline{d}}{4} \tag{2.41}$$

$$2\underline{x_1} = \frac{\underline{a}+\underline{b}+\underline{c}+\underline{d}}{2} = \left(\underline{a}+\underline{b}+\underline{c}+\underline{d}\right) + \mu_1\left(\underline{a}+\underline{b}+\underline{c}+\underline{d}\right)\left(\underline{d}-\underline{c}\right)^2 \tag{2.42}$$

$$\mu_1 = -\frac{1}{2\left(\underline{d}-\underline{c}\right)^2} \tag{2.43}$$

As we can see now that both the altitude and weight centers fit into this line, that proves our notable point also fits into the Euler line. This is the end of the proof. The center of circumscribed sphere $\underline{x_2}$ corresponds to μ_2 (A1. Appendix):

$$\underline{x_2} = \frac{1}{2}\left(\underline{a}+\underline{b}+\underline{c}+\underline{d}\right) \tag{2.44}$$

From (2.36):

$$2\underline{x_2} = \underline{a}+\underline{b}+\underline{c}+\underline{d} = \left(\underline{a}+\underline{b}+\underline{c}+\underline{d}\right) + \mu_2\left(\underline{a}+\underline{b}+\underline{c}+\underline{d}\right)\left(\underline{d}-\underline{c}\right)^2 \tag{2.45}$$

$$\mu_2 = 0 \tag{2.46}$$

Now we calculate the intersections of the Euler line with faces of the tetrahedron. In general, there are two such faces. But we can imagine that a special case is a vertex and the opposite face. Let us investigate this last situation now. Let the vertex be

$$\underline{x_3} = \underline{a} \tag{2.47}$$

From (2.34),

$$2\underline{a} = \left(\underline{a}+\underline{b}+\underline{c}+\underline{d}\right) + \mu_3\left(\underline{a}+\underline{b}+\underline{c}+\underline{d}\right)\left(\underline{d}-\underline{c}\right)^2 \tag{2.48}$$

In general, $\underline{a}$ is linearly dependent from $\underline{b}, \underline{c}$ and $\underline{d}$:

$$\underline{a} = \gamma_1\underline{b} + \gamma_2\underline{c} + \gamma_3\underline{d} = a_1\underline{i} + a_2\underline{j} + a_3\underline{k} \tag{2.49}$$

where $\underline{i}, \underline{j}, \underline{k}$ is the orthonormal system, and

$$a_1 = \gamma_1 b_1 + \gamma_2 c_1 + \gamma_3 d_1 \tag{2.50}$$

$$a_2 = \gamma_1 b_2 + \gamma_2 c_2 + \gamma_3 d_2 \tag{2.51}$$

$$a_3 = \gamma_1 b_3 + \gamma_2 c_3 + \gamma_3 d_3 \tag{2.52}$$

(2.49-51) is the way how to determine $\gamma_1, \gamma_2, \gamma_3$ from $\underline{a}, \underline{b}, \underline{c}, \underline{d}$.

From (2.38,2.48-49):

$$\underline{0} = \left(-(\gamma_1\underline{b} + \gamma_2\underline{c} + \gamma_3\underline{d}) + \underline{b}+\underline{c}+\underline{d}\right) + \mu_3\left(\gamma_1\underline{b} + \gamma_2\underline{c} + \gamma_3\underline{d} + \underline{b}+\underline{c}+\underline{d}\right)\left(\underline{d}-\underline{c}\right)^2 \tag{2.53}$$

Vectors $\underline{b}, \underline{c}$ and $\underline{d}$ are linearly independent:

$$0 = -\gamma_1 + 1 + \mu_3(\gamma_1 + 1)\left(\underline{d} - \underline{c}\right)^2 \tag{2.54}$$

$$0 = -\gamma_2 + 1 + \mu_3(\gamma_2 + 1)\left(\underline{d} - \underline{c}\right)^2 \tag{2.55}$$

$$0 = -\gamma_3 + 1 + \mu_3(\gamma_3 + 1)\left(\underline{d} - \underline{c}\right)^2 \tag{2.56}$$

$$\mu_3 = \frac{\gamma-1}{\gamma+1}\frac{1}{\left(\underline{d} - \underline{c}\right)^2} \text{ for } \gamma = \gamma_1 = \gamma_2 = \gamma_3 \neq -1 \tag{2.57}$$

Now we determine the point of intersection in general case. Let the face be bcd. The new notable point $\underline{x_4}$ fits now into bcd as $\underline{x_4}$ is the intersection of the Euler line and bcd:

$$\underline{x_4} = \underline{c} + \alpha\left(\underline{b} - \underline{c}\right) + \beta\left(\underline{c} - \underline{d}\right) \tag{2.58}$$

From (2.34,2.38,2.58):

$$2\left[\underline{c} + \alpha\left(\underline{b} - \underline{c}\right) + \beta\left(\underline{c} - \underline{d}\right)\right] = \left(\gamma_1\underline{b} + \gamma_2\underline{c} + \gamma_3\underline{d} + \underline{b} + \underline{c} + \underline{d}\right) + \mu_4\left(\gamma_1\underline{b} + \gamma_2\underline{c} + \gamma_3\underline{d} + \underline{b} + \underline{c} + \underline{d}\right)\left(\underline{d} - \underline{c}\right)^2 \tag{2.59}$$

Vectors $\underline{b}, \underline{c}, \underline{d}$ are linearly independent:

$$2\alpha = (\gamma_1 + 1)\left(1 + \mu_4\left(\underline{d} - \underline{c}\right)^2\right) \tag{2.60}$$

$$2 - 2\alpha + 2\beta = (\gamma_2 + 1)\left(1 + \mu_4\left(\underline{d} - \underline{c}\right)^2\right) \tag{2.61}$$

$$-2\beta = (\gamma_3 + 1)\left(1 + \mu_4\left(\underline{d} - \underline{c}\right)^2\right) \tag{2.62}$$

$$2 = (\gamma_1 + \gamma_2 + \gamma_3 + 3)\left(1 + \mu_4\left(\underline{d} - \underline{c}\right)^2\right) \tag{2.63}$$

$$\mu_4 = \frac{-1-(\gamma_1+\gamma_2+\gamma_3)}{\gamma_1+\gamma_2+\gamma_3+3}\frac{1}{\left(\underline{d} - \underline{c}\right)^2} \tag{2.64}$$

$$\underline{x_4} = \frac{1}{2}\left(\underline{a} + \underline{b} + \underline{c} + \underline{d}\right)\left(1 + \mu_4\left(\underline{d} - \underline{c}\right)^2\right) = \frac{1}{2}\left(\underline{a} + \underline{b} + \underline{c} + \underline{d}\right)\left(1 + \frac{-1-(\gamma_1+\gamma_2+\gamma_3)}{\gamma_1+\gamma_2+\gamma_3+3}\right) = \left(\underline{a} + \underline{b} + \underline{c} + \underline{d}\right)\frac{1}{\gamma_1+\gamma_2+\gamma_3+3} \tag{2.65}$$

Next is to calculate μ for the new notable point and investigate how many of them may exist.

3. Number of new notable points

Now we solve (2.4, 2.34)) for μ.

$$\left(\underline{x} - \underline{a}\right)^2 = \mu(\underline{b} \times \underline{c} + \underline{c} \times \underline{d} + \underline{d} \times \underline{b})^2 \tag{3.1}$$

$$2\underline{x} = \left(\underline{a} + \underline{b} + \underline{c} + \underline{d}\right) + \mu\left[\left(\underline{a} + \underline{b} + \underline{c} + \underline{d}\right)\left(\underline{d} - \underline{c}\right)^2\right] =$$

$$= \left(\underline{a} + \underline{b} + \underline{c} + \underline{d}\right)\left(1 + \mu\left(\underline{d} - \underline{c}\right)^2\right) \tag{3.2}$$

Following [4], we introduce new quantities, some are repeated from (2.17-19):

$$s = \left|\underline{a}\right|^2 + \left|\underline{b}\right|^2 + \left|\underline{c}\right|^2 + \left|\underline{d}\right|^2 \tag{3.3}$$

$$k_1 = \underline{ab} = \underline{cd} \tag{3.4}$$

$$k_2 = \underline{ac} = \underline{bd} \tag{3.5}$$

$$k_3 = \underline{ad} = \underline{bc} \tag{3.6}$$

Start simplifying:

$$\left(\underline{x} - \underline{a}\right)^2 = \left[\frac{\underline{a} + \underline{b} + \underline{c} + \underline{d}}{2}\left(1 + \mu\left(\underline{d} - \underline{c}\right)^2\right) - \underline{a}\right]^2 =$$

$$= \frac{1}{4}\left(s + 4(k_1 + k_2 + k_3)\right)\left(1 + \mu\left(\underline{d} - \underline{c}\right)^2\right)^2 + \left|\underline{a}\right|^2 - \left|\underline{a}\right|^2\left(1 + \mu\left(\underline{d} - \underline{c}\right)^2\right) -$$
$$-(k_1 + k_2 + k_3)\left(1 + \mu\left(\underline{d} - \underline{c}\right)^2\right) = \frac{1}{4}\left(s + 4(k_1 + k_2 + k_3)\right)\left(1 + \mu\left(\underline{d} - \underline{c}\right)^2\right)^2 -$$
$$-\left|\underline{a}\right|^2\left(\mu\left(\underline{d} - \underline{c}\right)^2\right) - (k_1 + k_2 + k_3)\left(1 + \mu\left(\underline{d} - \underline{c}\right)^2\right) \tag{3.7}$$

With (2.35):

$$\left(\underline{d} - \underline{c}\right)^2 = \frac{1}{2}\left[\left(\underline{b} - \underline{a}\right)^2 + \left(\underline{d} - \underline{c}\right)^2\right] = \frac{1}{2}(s - 4k_1) \tag{3.8}$$

Using (3.7-8):

$$\left(\underline{x} - \underline{a}\right)^2 = \frac{1}{4}\left(s + 4(k_1 + k_2 + k_3)\right)\left(1 + \frac{1}{2}\mu(s - 4k_1)\right)^2 - \left|\underline{a}\right|^2\left(\frac{1}{2}\mu(s - 4k_1)\right) -$$
$$(k_1 + k_2 + k_3)\left(1 + \frac{1}{2}\mu(s - 4k_1)\right) \tag{3.9}$$

Now the right side of (3.1) is expanded. A known identity:

$$\left(\underline{p}\ x\ \underline{r}\right).\left(\underline{s}\ x\ \underline{t}\right) = \left(\underline{p}.\underline{s}\right)\left(\underline{r}.\underline{t}\right) - \left(\underline{p}.\underline{t}\right)\left(\underline{r}.\underline{s}\right) \tag{3.10}$$

where notation dot means scalar (dot) product.

$$\left(\underline{b}\ x\ \underline{c} + \underline{c}\ x\ \underline{d} + \underline{d}\ x\ \underline{b}\right)^2 =$$
$$= \left(\underline{b}\ x\ \underline{c}\right).\left(\underline{b}\ x\ \underline{c}\right) + \left(\underline{b}\ x\ \underline{c}\right).\left(\underline{c}\ x\ \underline{d}\right) + \left(\underline{b}\ x\ \underline{c}\right).\left(\underline{d}\ x\ \underline{b}\right) +$$
$$+\left(\underline{c}\ x\ \underline{d}\right).\left(\underline{b}\ x\ \underline{c}\right) + \left(\underline{c}\ x\ \underline{d}\right).\left(\underline{c}\ x\ \underline{d}\right) + \left(\underline{c}\ x\ \underline{d}\right).\left(\underline{d}\ x\ \underline{b}\right) +$$
$$+\left(\underline{d}\ x\ \underline{b}\right).\left(\underline{b}\ x\ \underline{c}\right) + \left(\underline{d}\ x\ \underline{b}\right).\left(\underline{c}\ x\ \underline{d}\right) + \left(\underline{d}\ x\ \underline{b}\right).\left(\underline{d}\ x\ \underline{b}\right) \tag{3.11}$$

$$\left(\underline{b}\ x\ \underline{c}\right).\left(\underline{b}\ x\ \underline{c}\right) = \left(\underline{bb}\right)\left(\underline{cc}\right) - \left(\underline{bc}\right)^2 \tag{3.12}$$

$$\left(\underline{c}\ x\ \underline{d}\right).\left(\underline{c}\ x\ \underline{d}\right) = \left(\underline{cc}\right)\left(\underline{dd}\right) - \left(\underline{cd}\right)^2 \tag{3.13}$$

$$\left(\underline{d}\ x\ \underline{b}\right).\left(\underline{d}\ x\ \underline{b}\right) = \left(\underline{dd}\right)\left(\underline{bb}\right) - \left(\underline{db}\right)^2 \tag{3.14}$$

$$\left(\underline{b}\ x\ \underline{c}\right).\left(\underline{c}\ x\ \underline{d}\right) = \left(\underline{bc}\right)\left(\underline{cd}\right) - \left(\underline{db}\right)\left(\underline{cc}\right) \tag{3.15}$$

$$\left(\underline{b}\ x\ \underline{c}\right).\left(\underline{d}\ x\ \underline{b}\right) = \left(\underline{db}\right)\left(\underline{bc}\right) - \left(\underline{cd}\right)\left(\underline{bb}\right) \tag{3.16}$$

$$\left(\underline{c}\ x\ \underline{d}\right).\left(\underline{d}\ x\ \underline{b}\right) = \left(\underline{cd}\right)\left(\underline{db}\right) - \left(\underline{bc}\right)\left(\underline{dd}\right) \tag{3.17}$$

Using (3.4-6):

$$\left(\underline{b}\ x\ \underline{c}\right).\left(\underline{b}\ x\ \underline{c}\right) = \left(\underline{bb}\right)\left(\underline{cc}\right) - {k_3}^2 \tag{3.18}$$

$$\left(\underline{c}\ x\ \underline{d}\right).\left(\underline{c}\ x\ \underline{d}\right) = \left(\underline{cc}\right)\left(\underline{dd}\right) - {k_1}^2 \tag{3.19}$$

$$\left(\underline{d}\ x\ \underline{b}\right).\left(\underline{d}\ x\ \underline{b}\right) = \left(\underline{dd}\right)\left(\underline{bb}\right) - {k_2}^2 \tag{3.20}$$

$$\left(\underline{b}\ x\ \underline{c}\right).\left(\underline{c}\ x\ \underline{d}\right) = k_3 k_1 - k_2\left(\underline{cc}\right) \tag{3.21}$$

$$\left(\underline{b}\ x\ \underline{c}\right).\left(\underline{d}\ x\ \underline{b}\right) = k_2 k_3 - k_1\left(\underline{bb}\right) \tag{3.22}$$

$$\left(\underline{c}\ x\ \underline{d}\right).\left(\underline{d}\ x\ \underline{b}\right) = k_1 k_2 - k_3\left(\underline{dd}\right) \tag{3.23}$$

We introduce the new notation t:

$$t = \left(\underline{aa}\right)\left(\underline{bb}\right) + \left(\underline{bb}\right)\left(\underline{cc}\right) + \left(\underline{cc}\right)\left(\underline{dd}\right) + \left(\underline{dd}\right)\left(\underline{aa}\right) + \left(\underline{aa}\right)\left(\underline{cc}\right) + \left(\underline{bb}\right)\left(\underline{dd}\right) \tag{3.24}$$

With (3.11,3.18-23):

$$\left(\underline{b}\ x\ \underline{c} + \underline{c}\ x\ \underline{d} + \underline{d}\ x\ \underline{b}\right)^2 = \left|\underline{b}\right|^2\left|\underline{c}\right|^2 - {k_3}^2 + \left|\underline{c}\right|^2\left|\underline{d}\right|^2 - {k_1}^2 + \left|\underline{d}\right|^2\left|\underline{b}\right|^2 - {k_2}^2 +$$

$$+2\left[k_3 k_1 - k_2\left|\underline{c}\right|^2\right] + 2\left[k_2 k_3 - k_1\left|\underline{b}\right|^2\right] + 2\left[k_1 k_2 - k_3\left|\underline{d}\right|^2\right] =$$

$$= t - \left|\underline{a}\right|^2\left(s - \left|\underline{a}\right|^2\right) - {k_1}^2 - {k_2}^2 - {k_3}^2 +$$

$$+2\left[-k_2\left|\underline{c}\right|^2\right] + 2\left[-k_1\left|\underline{b}\right|^2\right] + 2\left[-k_3\left|\underline{d}\right|^2\right] \tag{3.25}$$

From (3.9) and (3.25):

$$\frac{1}{4}\left(s + 4(k_1 + k_2 + k_3)\right)\left(1 + \frac{1}{2}\mu(s - 4k_1)\right)^2 - \left|\underline{a}\right|^2\left(\frac{1}{2}\mu(s - 4k_1)\right) - (k_1 + k_2 + k_3)\left(1 + \frac{1}{2}\mu(s - 4k_1)\right) = \left[t - \left|\underline{a}\right|^2\left(s - \left|\underline{a}\right|^2\right) - {k_1}^2 - {k_2}^2 - {k_3}^2 + 2\left[-k_2\left|\underline{c}\right|^2 - k_1\left|\underline{b}\right|^2 - k_3\left|\underline{d}\right|^2\right]\right] \tag{3.26}$$

From (3.26) follows that the number of new notable points is two, one or zero.

First, from (3.8):

$$s - 4k \neq 0 \tag{3.27}$$

otherwise $\underline{c}$ and $\underline{d}$ would coincide. Another possibility to reduce the number of solutions to 1 is the following:

$$s + 4(k_1 + k_2 + k_3) = 0 \tag{3.28}$$

From which it follows, (3.3-6):

$$\underline{a} + \underline{b} + \underline{c} + \underline{d} = \underline{0} \tag{3.29}$$

That is, weight point and circumcenter coincide. This is the regular tetrahedron. In case of a regular tetrahedron, the new notable point coincides the center of the tetrahedron. In this case, number of new notable points equals to 1.

From [4] we can see a special case when the triangle has a 90 deg angle, and then there is one new notable point that coincides with the vertex at this angle. In that case, the center of the circumscribed circle fits into the section opposite to this vertex, Thales circle. In our present case, the corresponding situation is when the center of the circumscribed sphere coincides with the intersection of the Euler line with a face of the tetrahedron.

In (2.44), the center of the circumscribed sphere corresponds to

$$\mu_2 = 0 \tag{3.30}$$

In (2.64), intersection of the Euler line with one face is at

$$\mu_4 = \frac{-1-(\gamma_1+\gamma_2+\gamma_3)}{\gamma_1+\gamma_2+\gamma_3+3}\frac{1}{(\underline{b}-\underline{a})^2} \tag{3.31}$$

From $\mu_2 = \mu_4$,

$$\gamma_1 + \gamma_2 + \gamma_3 = -1 \tag{3.32}$$

Now we go back to (2.4) defining μ:

$$\left(\underline{x_2} - \underline{a}\right)^2 = \mu_2(\underline{b}\ x\ \underline{c} + \underline{c}\ x\ \underline{d} + \underline{d}\ x\ \underline{b})^2 \tag{3.33}$$

From (3.30) and (3.33),

$$\underline{x_2} = \underline{a} \tag{3.34}$$

In words, when the circumcenter fits into a face, then the new notable point is the vertex opposite to the face. This is analogous to the 2-dimensional case [4]. We have analyzed this case above in (2.57) and concluded that

$$\gamma = \gamma_1 = \gamma_2 = \gamma_3 \tag{3.35}$$

From (3.32) and (3.35):

$$\gamma = -\frac{1}{3} \tag{3.36}$$

that is related to the bond angles in methane molecule. In case of identical vector lengths, $\cos\varphi = -\frac{1}{3}$ where φ is the bond angle. Other cases are left for another publication.

4. Further generalization possibilities

The suggested general formulae for the notable point:

$$\frac{[m_1(x-x_1)]^2}{[m_2(x_2,\dots x_n)]^2} = \frac{[m_1(x-x_2)]^2}{[m_2(x_3,\dots x_n,x_1)]^2} = \cdots = \frac{[m_1(x-x_n)]^2}{[m_2(x_1,\dots x_{n-1})]^2} \tag{4.1}$$

where m_1, m_2 are some measures, not necessarily the same. This is also suitable for the case when number of vertices is greater than 4 for 3 dimensions, and for higher dimensional cases.

5. Conclusions

Our previously introduced notable point of plane triangles has been extended for tetrahedrons. We followed the lines of [4], with the details changed. Main statements are summarized here.

- (2.2) generalizes the definition of the new notable point for tetrahedrons.
- (2.31) says that the new notable point is on a line on which the center of circumscribed sphere fits into.
- If the tetrahedron has two non-intersecting edges that meet conditions (2.35) and (2.37-38), then the new notable point fits into the Euler line, (2.36).
- (3.26) says that the number of new notable points is two, one or zero.
- Exactly one new notable point exists for a regular tetrahedron, that is identical to the center of the tetrahedron, (3.29).
- Exactly one new notable point exists for a tetrahedron whose center of circumscribed sphere fits into the boundary of the tetrahedron, (3.34). This case, the new notable point is the vertex, opposite to the face on which the circumcenter is placed.
- Definition of the new notable point is suggested for n dimensions, (4.1).

We left open the problems of two and zero solutions, and an example for more than 3-dimensional case and applications. These are intended to be detailed in a forthcoming publication. Applications may be expected in circuits and systems theory and in quantum communications.

This paper can be interesting for physicists and chemists because there are some molecules having tetrahedral molecular geometry.

6. Acknowledgments

Not long ago this author initiated a research on notable points from practical aspects. It is a nice memory that two mathematicians, Prof. K. Bezdek and Prof. V. Zoller were interested. Our common work in the topic of Euler line of plane triangles is a solid basis of this research as well, and the author is grateful for them.

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7. References

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A1. Appendix: Circumcenter of a tetrahedron

Here we express the circumcenter $\underline{x_2}$ of a tetrahedron abcd when the origin is the Monge point.

$$(\underline{x_2} - \underline{a})^2 = (\underline{x_2} - \underline{b})^2 \tag{A1.1}$$

$$(\underline{x_2} - \underline{b})^2 = (\underline{x_2} - \underline{c})^2 \tag{A1.2}$$

$$(\underline{x_2} - \underline{c})^2 = (\underline{x_2} - \underline{d})^2 \tag{A1.3}$$

From (A1.1):

$$\left(2\underline{x_2} - \underline{a} - \underline{b}\right)\left(\underline{a} - \underline{b}\right) = 0 \tag{A1.4}$$

$$\underline{a} \neq \underline{b} \tag{A1.5}$$

$$2\underline{x_2} - \underline{a} - \underline{b} + \underline{z} = \underline{0} \tag{A1.6}$$

$$\underline{z}\left(\underline{a} - \underline{b}\right) = 0 \tag{A1.7}$$

From (2.13), characterizing the Monge point as origin:

$$\underline{z} = \alpha\left(\underline{c} + \underline{d}\right) \tag{A1.8}$$

is a possible solution. (A1.6):

$$2\underline{x_2} - \underline{a} - \underline{b} + \alpha\left(\underline{c} + \underline{d}\right) = \underline{0} \tag{A1.9}$$

Vertices are permuted:

$$2\underline{x_2} - \underline{c} - \underline{d} + \beta\left(\underline{a} + \underline{b}\right) = \underline{0} \tag{A1.10}$$

From (A1.9-10):

$$-\underline{a} - \underline{b} + \alpha\left(\underline{c} + \underline{d}\right) + \underline{c} + \underline{d} - \beta\left(\underline{a} + \underline{b}\right) = \underline{0} \tag{A1.11}$$

Vectors $\left(\underline{a} + \underline{b}\right)$ and $\left(\underline{c} + \underline{d}\right)$ are linearly independent:

$$\alpha = \beta = -1 \tag{A1.12}$$

Then from (A1.9):

$$\underline{x_2} = \tfrac{1}{2}\left(\underline{a} + \underline{b} + \underline{c} + \underline{d}\right) \tag{A1.13}$$

A.2. Appendix: Conditions (2.27-28)

$$\left(\underline{c} - \underline{d}\right) x \left(2\underline{c}\ x\ \underline{d}\right) = 2\underline{c}\left(\left(\underline{c} - \underline{d}\right)\underline{d}\right) - \underline{d}\left(\left(\underline{c} - \underline{d}\right)2\underline{c}\right) \tag{A2.1}$$

Expanding (2.27):

$$\left(\underline{c}-\underline{d}\right)\,x\,\left(2\underline{c}\,x\,\underline{d}\right)=\alpha_1\left(\underline{c}+\underline{d}\right)+\alpha_1'\left(\underline{c}-\underline{d}\right) \quad \text{(A2.2)}$$

New notations:

$$2\left(\underline{c}-\underline{d}\right)\underline{d}=\alpha \quad \text{(A2.3)}$$

$$\left(\underline{c}-\underline{d}\right)2\underline{c}=\beta \quad \text{(A2.4)}$$

$$\alpha_1+\alpha_1'=\ \alpha \quad \text{(A2.5)}$$

$$-\alpha_1+\alpha_1'=\ \beta \quad \text{(A2.6)}$$

$$\alpha_1'=\left(\underline{c}-\underline{d}\right)\underline{d}+\left(\underline{c}-\underline{d}\right)2\underline{c}=0 \quad \text{(A2.7)}$$

$$\left|\underline{c}\right|^2=\left|\underline{d}\right|^2 \quad \text{(A2.8)}$$

$$\left(\underline{c}-\underline{d}\right)\,x\,\left(2\underline{c}\,x\,\underline{d}\right)=\alpha_1\left(\underline{c}+\underline{d}\right) \quad \text{(A2.9)}$$

Similarly, for (2.28):

$$\left|\underline{a}\right|^2=\left|\underline{b}\right|^2 \quad \text{(A2.10)}$$

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Dr. János Ladvánszky is a Hungarian electrical engineer. He has been retired from Ericsson Hungary. His topics are circuits and systems theory, with microwave and optical applications, and recently, quantum communications. He has more than 180 journal and conference papers, 14 patents and 6 books. He is a Senior Member of IEEE. He is available on LinkedIn, ResearchGate and Facebook.

His mathematical activity is related to applications. His mathematics-related contributions include a new notable point for plane triangles, with co-authors, applied for microwave measurements, an exchange identity for inner products containing Volterra kernels, applied in weakly nonlinear power matching, application of the Moore-Penrose generalized inverse for MIMO (Multiple Input, Multiple Output) reception and for identification of lumped element circuit models of electrical devices, application of the concept of equivalence for nonlinear circuits, avoiding analytical solution of nonlinear equations in mixer problems, a Bessel function method for calculating intermodulation, application of an identity for Hilbert transformation of the product of two functions in modulation problems, analytical determination of the output spectrum of a noisy oscillator, application of matrix calculus in explanation of the adjoint network method for circuit optimization, application of the Routh-Hurwitz test for stability of nonlinear tuned amplifiers, reversal of the recursion in calculating the determinant of tridiagonal matrices, applied for stability of linear multistage amplifiers, being a reviewer for a book on matroids and their applications in statics and circuit theory.

His activity in physics is within the area of electrical engineering and quantum physics that are the main topics of the author. These contributions include two-port describing functions, exotic phenomena in tuned nonlinear circuits, modelling of optically illuminated semiconductor devices, a channel allocation for optical communications to decrease the effects of harmonics and intermodulation products, theoretically not bounded large signal properties of mixers, a new phase detector circuit, physical basics of violating causality, an efficient way for noise reduction in digital communications, frequency limitations for digital modulation. More recently, an overview was presented on quantum entanglement, a book, another book on power matching, and some results in quantum communications that haven't been published yet, such as measurement of the speed of quantum entanglement.