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The Euler Line Revisited

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1 Introduction

Recent research on microwave reflectometers in electrical engineering [1] led naturally to the following geometric extremum problem. A reflectometer can be described by the equation

$$y = \alpha + \frac{\beta \cdot x}{1 - \gamma \cdot x}, \tag{1}$$

Der Beitrag beginnt mit der Beschreibung eines konkreten technischen Problems und seiner mathematischen Formulierung. Statt die zugehörige Fragestellung amschliessend einfach zu lösen, gehen die Autoren den Verbindungen innerhalb der Mathematik genauer nach: Sie führen in diesem Fall — überraschenderweise — in die klassische Dreiecksgeometrie, zur *Euler-Geraden*. Ein schönes Beispiel für die befruchtenden Wechselwirkungen zwischen der Mathematik und ihren Anwendungen. *ust*

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where x denotes the reflection coefficient of the measured device, y is the value read off the reflectometer, and α, β and γ represent the imperfections of the measuring instrument. All these quantities are complex numbers.

Equation (1) expresses the fact that the actual and the measured parameters usually differ due to the imperfections in the reflectometer. Consequently, the parameters α, β and γ have to be determined before the measurement. This step is carried out using (three) devices whose reflection coefficients x_i, $i = 1,2,3$ are assumed to be known, and who yield values y_i, $i = 1,2,3$. Thus, the pairs (x_i, y_i), $i = 1,2,3$ can be used to eliminate α, β and γ of (1). We easily obtain

$$\frac{y-y_1}{y-y_2} \cdot \frac{y_3-y_2}{y_3-y_1} = \frac{x-x_1}{x-x_2} \cdot \frac{x_3-x_2}{x_3-x_1}\,. \tag{2}$$

In practice the reflection coefficients of the above mentioned known devices may differ slightly from the assumed values, resulting in a measurement error. One possibility for reducing this effect is to keep the sensitivities expressed by $(3a-c)$ small. The equations $(3a-c)$ are derived from equation (2):

$$\left|\frac{dx}{dx_1}\right| = \left|\frac{x-x_2}{x_1-x_3} \cdot \frac{x-x_3}{x_1-x_2}\right|, \tag{3a}$$

$$\left|\frac{dx}{dx_2}\right| = \left|\frac{x-x_3}{x_2-x_1} \cdot \frac{x-x_1}{x_2-x_3}\right|, \tag{3b}$$

$$\left|\frac{dx}{dx_3}\right| = \left|\frac{x-x_1}{x_3-x_2} \cdot \frac{x-x_2}{x_3-x_1}\right|. \tag{3c}$$

A possible way of minimizing the sensitivities is to make them equal and then to find the minimum:

$$\left|\frac{dx}{dx_1}\right| = \left|\frac{dx}{dx_2}\right| = \left|\frac{dx}{dx_3}\right|. \tag{4}$$

The same minimum can be achieved by minimizing the sum of the sensitivities of $(3a-c)$. Thus, the equations $(3a-c)$ and (4) yield

$$\left|\frac{x-x_1}{x_2-x_3}\right| = \left|\frac{x-x_2}{x_3-x_1}\right| = \left|\frac{x-x_3}{x_1-x_2}\right|. \tag{5}$$

After a change in the notation and by using planar vectors instead of complex numbers we can formulate our result about the Euler line of triangles as follows. In the forthcoming discussion XY denotes the closed line segment determined by the points X and Y of the (Euclidean) plane and also the length of this segment.

Theorem *Let ABC be a triangle with vertices A, B and C. If P is a point in the plane of ABC with the property that $\frac{PA}{BC} = \frac{PB}{AC} = \frac{PC}{AB}$, then P is collinear with the centroid, the orthocenter and the center of the circumscribed circle of ABC; that is, P lies on the Euler line of ABC. Moreover, the number of points P with $\frac{PA}{BC} = \frac{PB}{AC} = \frac{PC}{AB}$ is 2 or 1 or 0 according to whether ABC is, respectively, an acute triangle different from a regular triangle, a regular triangle or a right triangle, or an obtuse triangle. Finally, $\frac{PA}{BC} = \frac{PB}{AC} = \frac{PC}{AB} \geq \frac{1}{\sqrt{3}}$ with equality only if ABC is a regular triangle.*

2 Proof of the Theorem

We choose the origin to be at the orthocenter of ABC. However, we remark here that an analogue of the following argument also works in the case that the origin is the center of the circumscribed circle of ABC.

As usual the position vector of the point X is denoted by $\underline{x}$. Moreover, the inner product of the vectors $\underline{x}$ and $\underline{y}$ is denoted by $\underline{xy}$.

By the choice of the origin we have

$$\underline{a}(\underline{b}-\underline{c}) = \underline{b}(\underline{c}-\underline{a}) = \underline{c}(\underline{a}-\underline{b}) \,, \tag{6}$$

that is

$$\underline{ab} = \underline{bc} = \underline{ca} \,. \tag{7}$$

Let P be a point of the plane such that $\frac{PA}{BC} = \frac{PB}{AC} = \frac{PC}{AB} = \sqrt{\mu}$ for some $\mu > 0$. This implies that

$$(\underline{p}-\underline{a})^2 = \mu \cdot (\underline{b}-\underline{c})^2 \,, \tag{8a}$$

$$(\underline{p}-\underline{b})^2 = \mu \cdot (\underline{c}-\underline{a})^2 \,, \tag{8b}$$

$$(\underline{p}-\underline{c})^2 = \mu \cdot (\underline{b}-\underline{a})^2 \,. \tag{8c}$$

$(8a), (8b)$ and $\underline{c}(\underline{a}-\underline{b}) = 0$ easily yield

$$\underline{p} = \frac{\mu+1}{2} \cdot (\underline{a}+\underline{b}) + \lambda_1 \cdot \underline{c} \tag{9a}$$

for some real λ_1.

By symmetry we get

$$\underline{p} = \frac{\mu+1}{2} \cdot (\underline{b}+\underline{c}) + \lambda_2 \cdot \underline{a} \tag{9b}$$

for some real λ_2.

From $(9a)$ and $(9b)$ it follows that $\left(\frac{\mu+1}{2} - \lambda_2\right) \cdot \underline{a} - \left(\frac{\mu+1}{2} - \lambda_1\right) \cdot \underline{c} = \underline{o}$. Hence, since $\underline{a}$ and $\underline{c}$ are linearly independent,

$$\lambda_1 = \lambda_2 = \frac{\mu+1}{2} \,. \tag{10}$$

Thus, $(9a)$ can be written in the form

$$\underline{p} = \frac{\mu+1}{2}(\underline{a}+\underline{b}+\underline{c}) \,. \tag{11}$$

It is easy to check that the position vectors of the orthocenter, the centroid and the center of the circumscribed circle of the triangle ABC are $\underline{o}$, $\frac{1}{3} \cdot (\underline{a}+\underline{b}+\underline{c})$ and $\frac{1}{2} \cdot (\underline{a}+\underline{b}+\underline{c})$.

Hence, because of (11), $\underline{p}$ is collinear with the above points; that is, it lies on the Euler line of the triangle ABC.

Now we determine the number of points P for which $\frac{PA}{BC} = \frac{PB}{AC} = \frac{PC}{AB} = \sqrt{\mu}$ with some $\mu > 0$. Let $s = \underline{a}^2 + \underline{b}^2 + \underline{c}^2$ and $k = \underline{a}\underline{b} = \underline{b}\underline{c} = \underline{c}\underline{a}$. Using (6), (7), (8$a$) and (11) a rather simple computation shows that μ satisfies the equation

$$(12) \qquad (s + 6k) \cdot \mu^2 + 2 \cdot (6k - s) \cdot \mu + (s - 2k) = 0 \ .$$

If $s + 6k = 0$, then $s + 6k = (\underline{a} + \underline{b} + \underline{c})^2$ implies that the orthocenter and the centroid of the triangle ABC coincide. That is, ABC is a regular triangle and we get $\mu = \frac{1}{3}$ from (12). Thus, we assume that $s + 6k \neq 0$, hence that ABC is not a regular triangle. The discriminant D of (12) is

$$(13) \qquad D = -32k \left((\underline{a} - \underline{b})^2 + (\underline{b} - \underline{c})^2 + (\underline{c} - \underline{a})^2 \right) \ .$$

Hence, (12) has respectively two, one and no real solutions for acute (non-regular), right and obtuse triangles. However, we have to show that any solution of (12) is non-negative. We prove this by showing that the minimum value of the solutions of (12) is $\frac{1}{3}$. Namely, from (12) we get

$$(14) \qquad \mu = \frac{s - 6k \pm \sqrt{-16k(s - 3k)}}{s + 6k} \ .$$

A simple computation shows that $\mu \geq \frac{1}{3}$ is equivalent to $s + 6k \geq 0$. As we have seen above, equality is attained here only if the triangle ABC is regular. This completes the proof of the Theorem. □

References

[1] J. Ladvánszky, K. Bezdek, A. Hilt and V. Zoller, *Minimum sensitivity calibration for reflectometers*, 2nd International Workshop on Integrated Non-Linear Microwave and Millimeterwave Circuits, Oct. 7–9, 92, Duisburg Univ. (accepted for publication), pp. 1–3.

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