PROJECTILE MOTION *Time of Flight*

The time the object is in motion is called the "time of flight" (TOF). This is the time from its "launch" until it reaches the ground. We can always define the coordinate system such that the final position on the ground is at y = 0. The motion uses the y(t) kinematic equation, since this will determine the TOF; the x(t) has no effect on this. Essentially, the projectile TOF is that of an object moving vertically up and then back down- how long does this take? The x-motion is irrelevant, for now. Start with y(t):

$$y(t) = y_0 + v_0\, t \sin(\theta) - \frac{1}{2}\, g\, t^2 \tag{1}$$

At the TOF, t = T, and y(T) is zero, so that we have

$$0 = y_0 + v_0\, T \sin(\theta) - \frac{1}{2}\, g\, T^2$$

In the general case, the initial y is not zero, so we must use the quadratic solution, with

$$a = \frac{-1}{2}\, g \qquad b = v_0 \sin(\theta) \qquad c = y_0$$

This leads to

$$\boxed{T = \frac{1}{g}\left[v_0 \sin(\theta) + \sqrt{\left(v_0 \sin(\theta)\right)^2 + 2\, g\, y_0}\right]} \tag{2}$$

The same result is obtained if we find the time for the object to rise to its maximum height, and then add the time it takes to fall from that height to the ground. Here we use the fact that the y-velocity is zero at the top of the trajectory, so we find the "rise" time τ using

$$v_y(t) = v_0 \sin(\theta) - g\, t \qquad 0 = v_0 \sin(\theta) - g\, \tau \qquad \boxed{\tau = \frac{v_0 \sin(\theta)}{g}}$$

and then we can find the maximum height with

$$y_{max} = y(\tau) = y_0 + v_0\, \tau \sin(\theta) - \frac{1}{2}\, g\, \tau^2$$

which will be

$$\boxed{y_{max} = y_0 + \frac{\left(v_0 \sin(\theta)\right)^2}{2\, g}}$$

It will then take a time γ to fall **from rest** at this height to y = 0, so that

$$0 = y_{max} + 0 - \frac{1}{2}\, g\, \gamma^2 \qquad \gamma = \sqrt{\frac{2\, y_{max}}{g}}$$

Then the total TOF is

$$T = \tau + \gamma = \frac{v_0 \sin(\theta)}{g} + \sqrt{\frac{2\, y_{max}}{g}} = \frac{v_0 \sin(\theta)}{g} + \sqrt{\frac{2}{g}\left[y_0 + \frac{\left(v_0 \sin(\theta)\right)^2}{2\, g}\right]}$$

When simplified this is the same as Eq(2). Note that for zero initial height, $\tau = \gamma$ (rise time = fall time).

Next, consider some special cases of the TOF. First, if the initial y is zero,

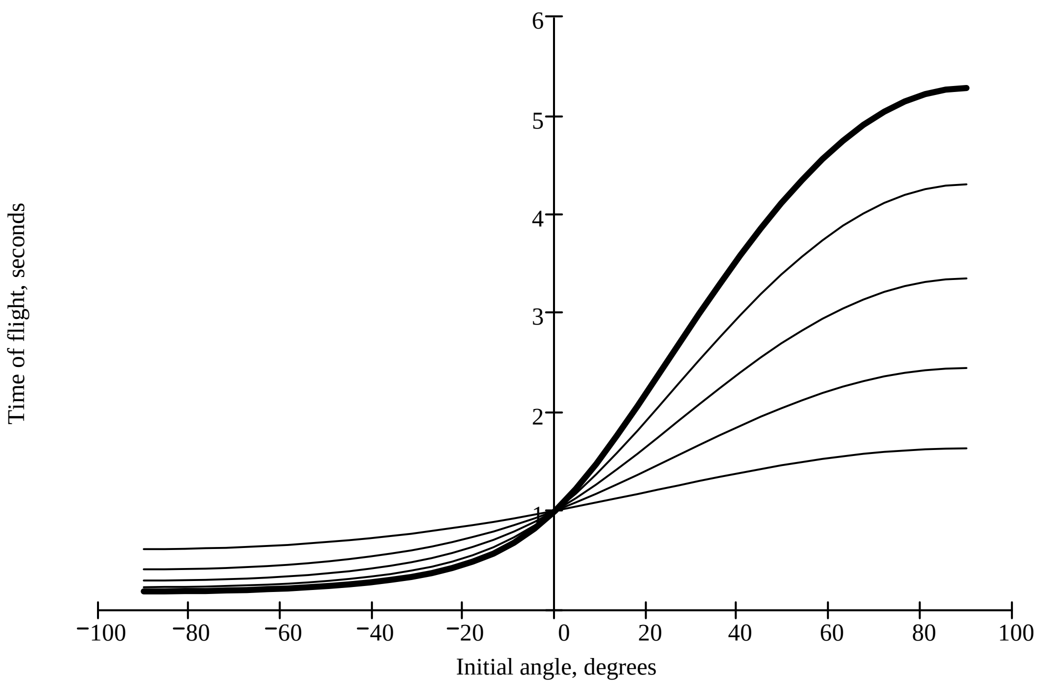
$$\boxed{T = \frac{2\, v_0}{g}\, \sin(\theta)} \qquad y_0 = 0$$

$$T = \frac{2\, v_0}{g} \qquad y_0 = 0 \qquad \theta = \frac{\pi}{2}$$

Note that this TOF is twice the time to maximum height τ found above. If the initial angle is zero (we must be at a positive initial height):

$$T = \sqrt{\frac{2\, y_0}{g}} \qquad \theta = 0$$

We can make various graphs of the TOF vs its parameters; one interesting plot is TOF vs. initial angle, for a given initial height (nonzero), with initial velocity as a parameter.



In this plot the initial height is 5 m, and the velocity is 5, 10, 15, 20, 25 m/s, with the thick line being 25 m/s. At an initial angle of zero degrees the initial velocity is purely horizontal; note that all the TOF's are the same, one second, since this is the time for an object to fall 5 m from rest. As the object is thrown downward (negative angles), the TOF becomes shorter than the free-fall time (one second). In the limit of a direct downward throw (minus 90 degrees), as the velocity becomes large,

$$\lim_{v_0 \to \infty} \frac{1}{g}\left[v_0 \sin\left(\frac{-\pi}{2}\right) + \sqrt{\left(v_0 \sin\left(\frac{-\pi}{2}\right)\right)^2 + 2\, g\, y_0} \right] = 0$$

which says that the object reaches the ground very quickly. This condition obtains when the first term in the radical dominates the second term.