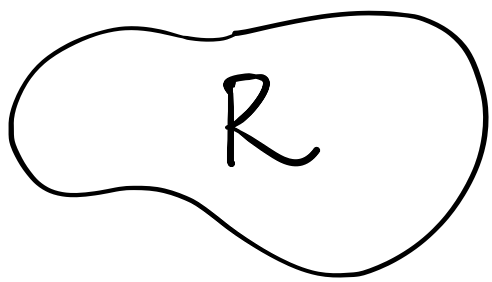
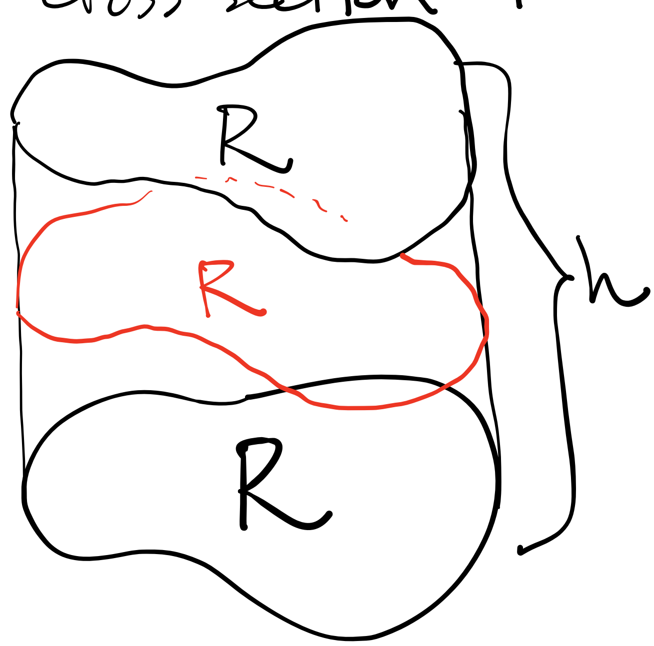
Class 4 - §6.2

Volume

Basic Idea: Say R is a region in the plane:



- construct a height h "cylinder" with cross section R

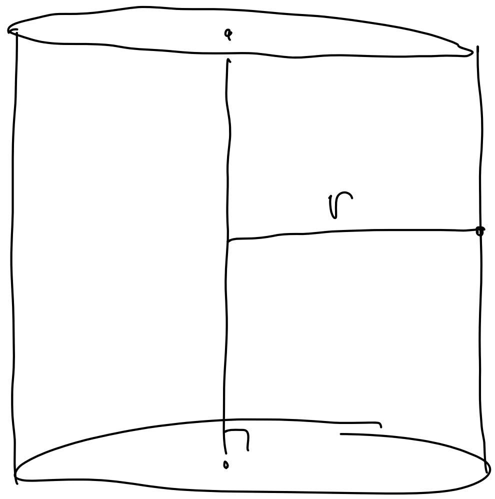


FACT: This thing has volume

area(R)$\cdot$ h

Ex:

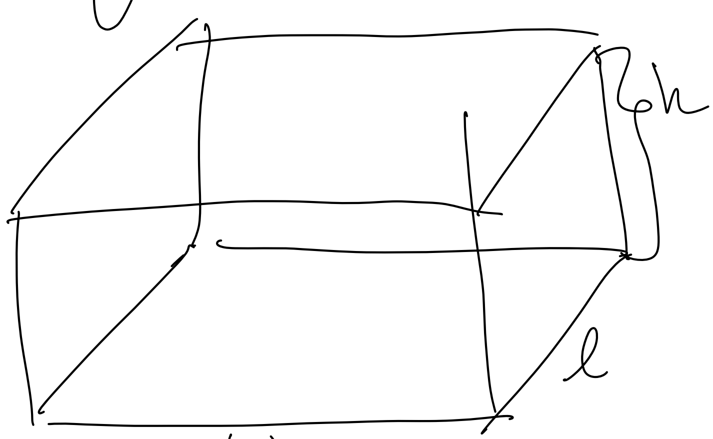
R = circle of radius r



volume = area(R)$\cdot$h

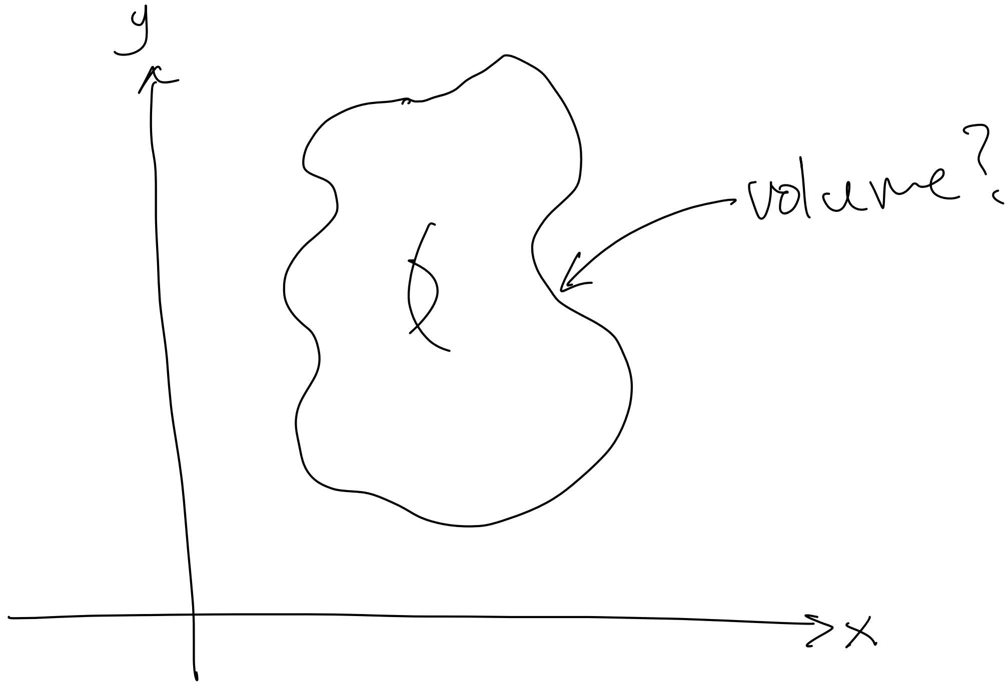
$= \pi r^2 \cdot h$

R = rectangle with legs of length l and w



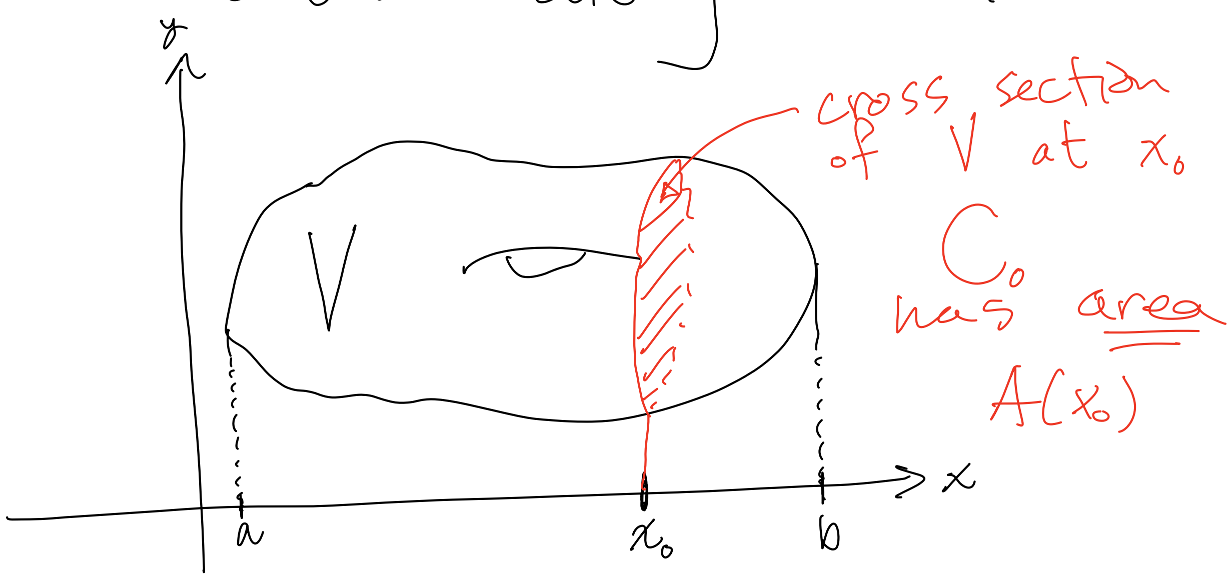
volume = $l \times w \times h$

What about things that aren't "cylinders?"



Intuition

- take your solid and lay it down along the x-axis



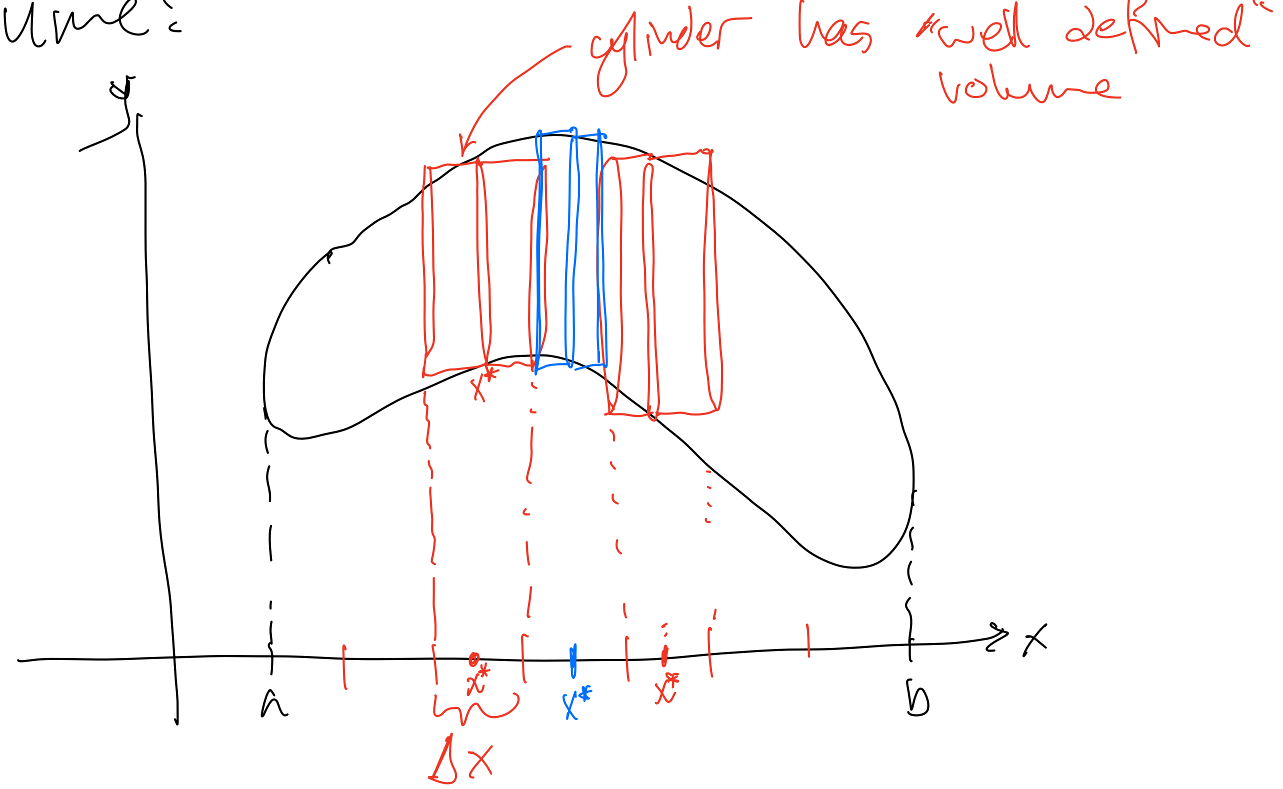
- consider cross-sections

i.e. for x_0, say C_0, write $A(x_0)$ for the area of C_0

Define volume of V to be

$$\int_a^b A(x)\,dx$$

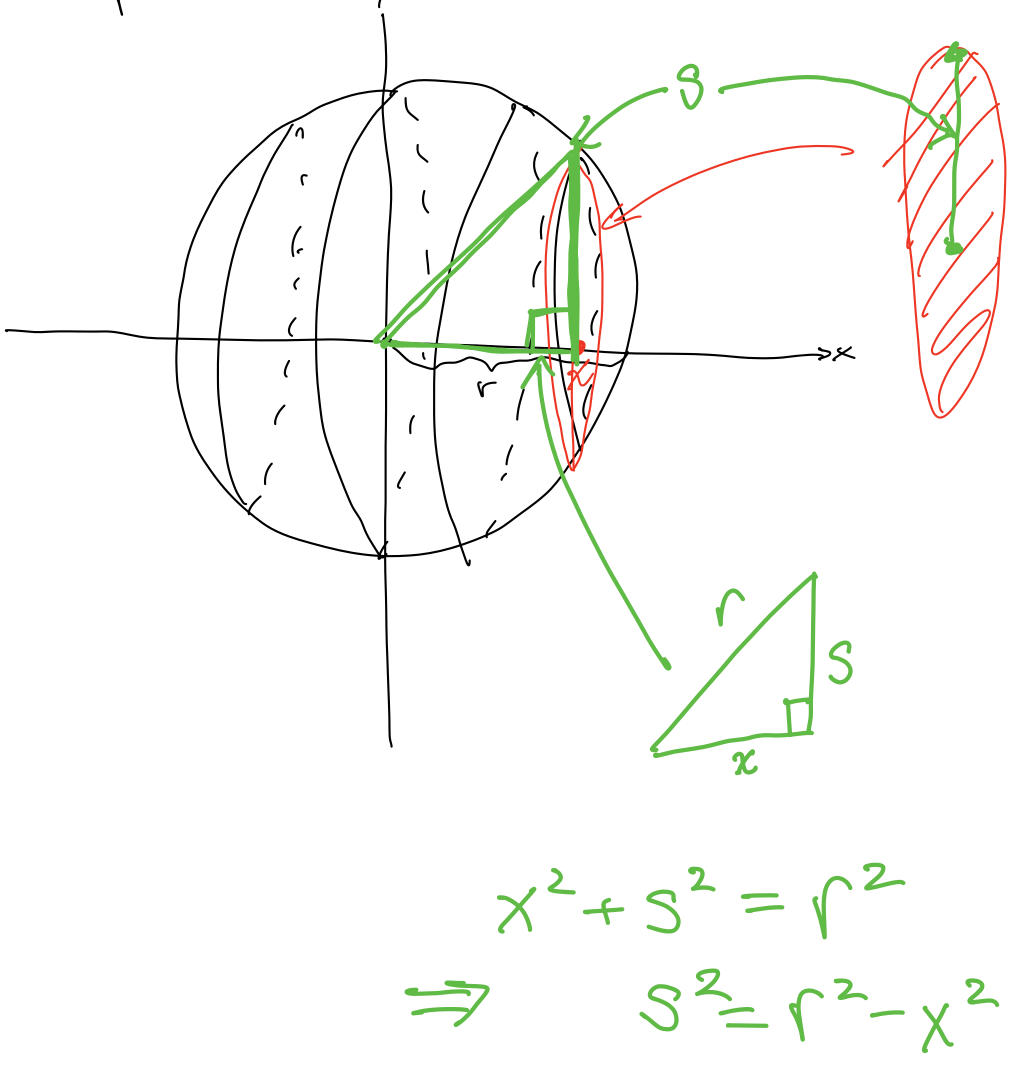
Why should this work?
Why should this behave like "volume?"



$$\text{Vol} \approx \sum_{i=1}^{n} A(x_i^*)\cdot\Delta x$$

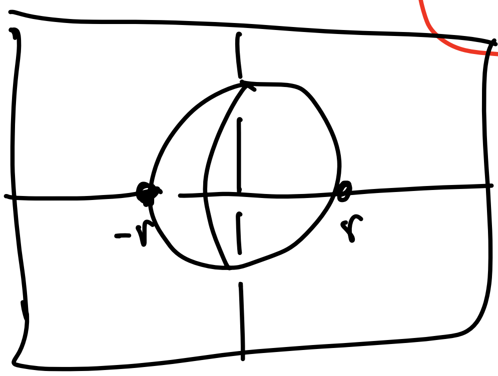
$$n\to\infty \rightsquigarrow \int_a^b A(x)\,dx = \text{volume}$$

Ex: Compute the volume of the sphere of radius r



$$x^2+s^2=r^2$$
$$\Rightarrow \quad s^2=r^2-x^2$$
$$\Rightarrow \quad s=\sqrt{r^2-x^2}$$

So $A(x)=\pi s^2=\pi\left(\sqrt{r^2-x^2}\right)^2=\pi(r^2-x^2)$



cross-sectional area of the sphere of radius r

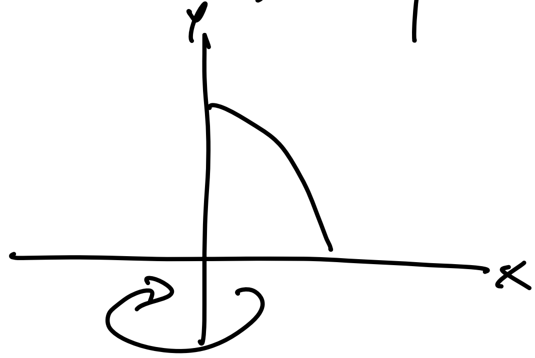
$\Rightarrow$ volume is

$$\int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi (r^2 - x^2)\,dx$$

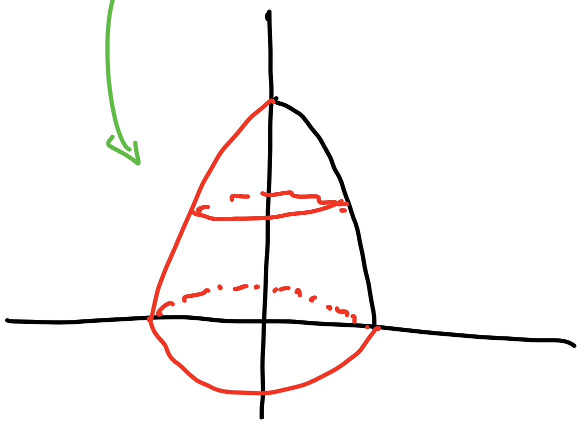
$$= \cdots = \frac{4}{3}\pi r^3$$

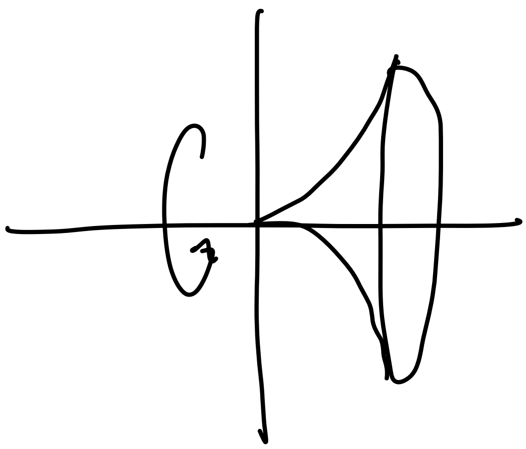
Solids of revolution

Idea: Start with a curve in the first quadrant

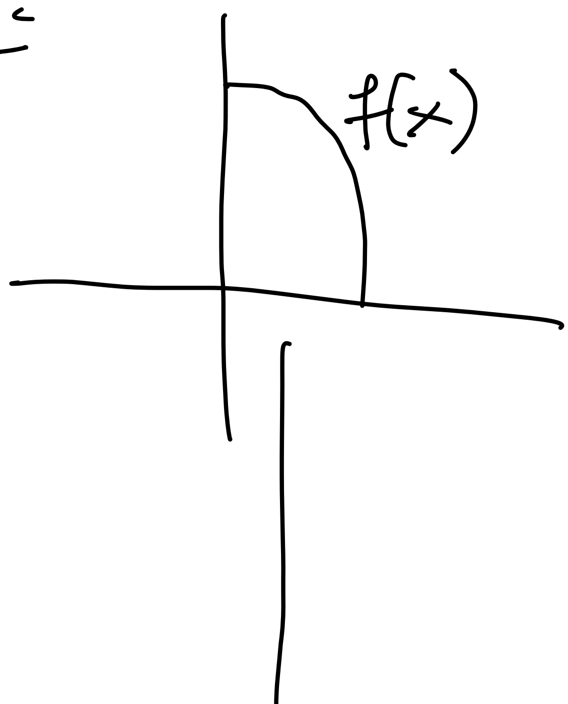


and "revolve" or "rotate" it around the y-axis or x-axis

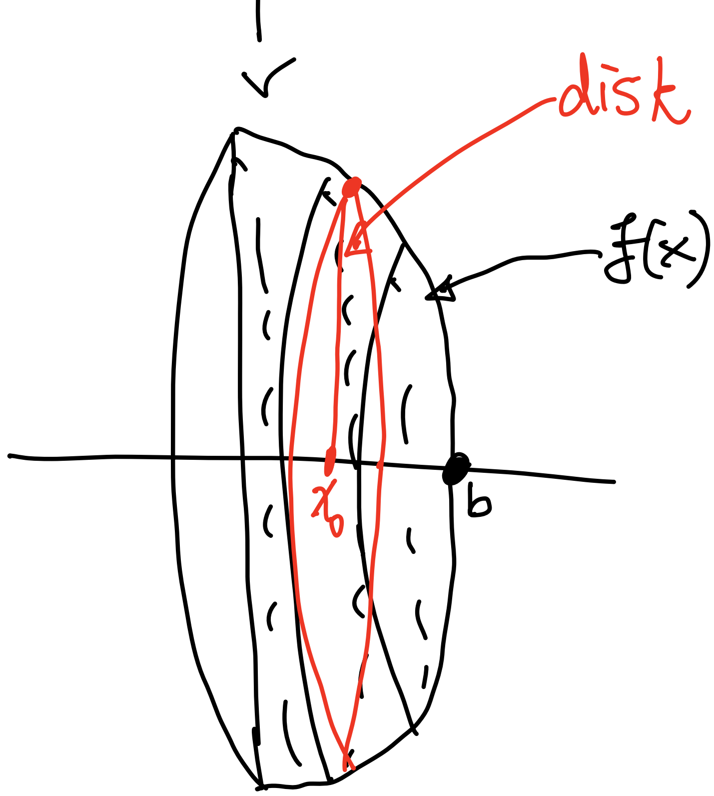




Ex:



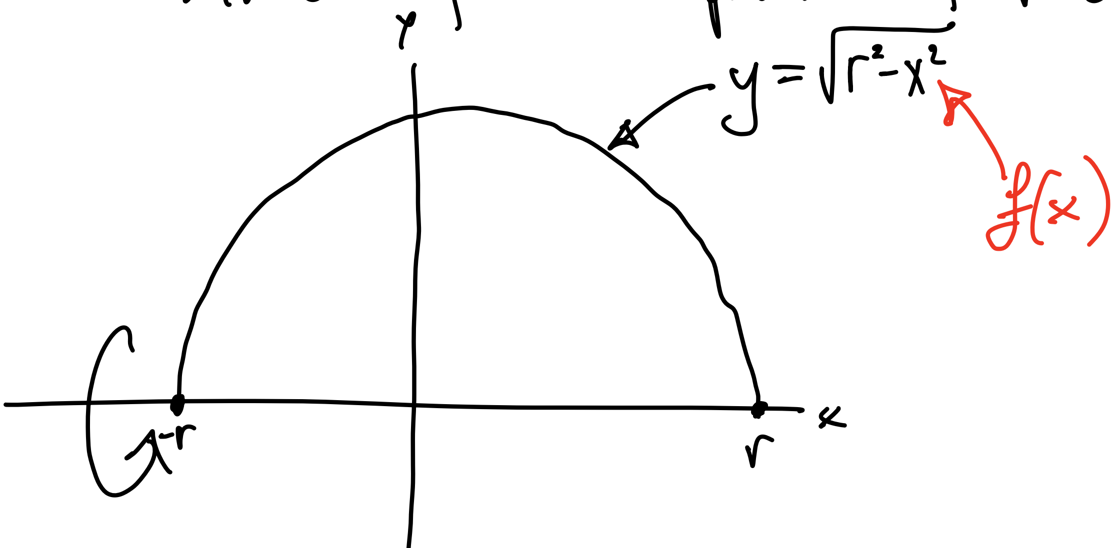
rotate around the x-axis



This disk has radius $f(x_0)$, hence area $\pi f(x_0)^2$

$$\text{volume} = \int_0^b \pi f(x)^2 \, dx$$

<u>Ex</u>: Volume of a sphere of rad. r:



revolve around x-axis to get sphere, so volume is

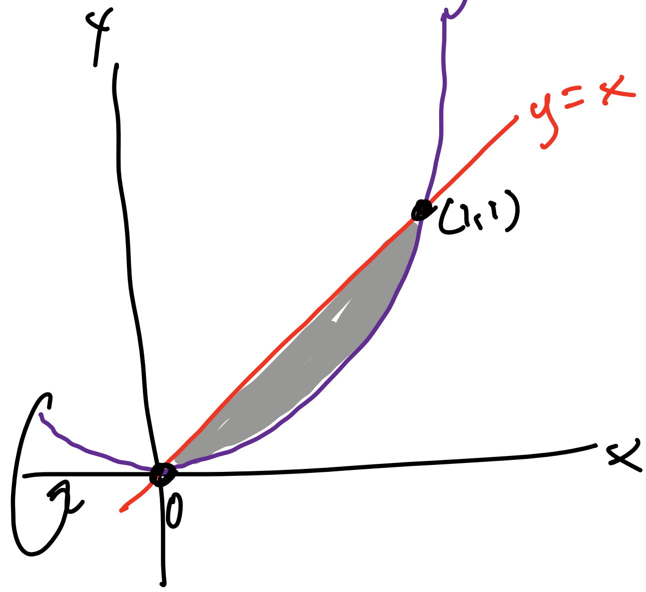
$$\int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

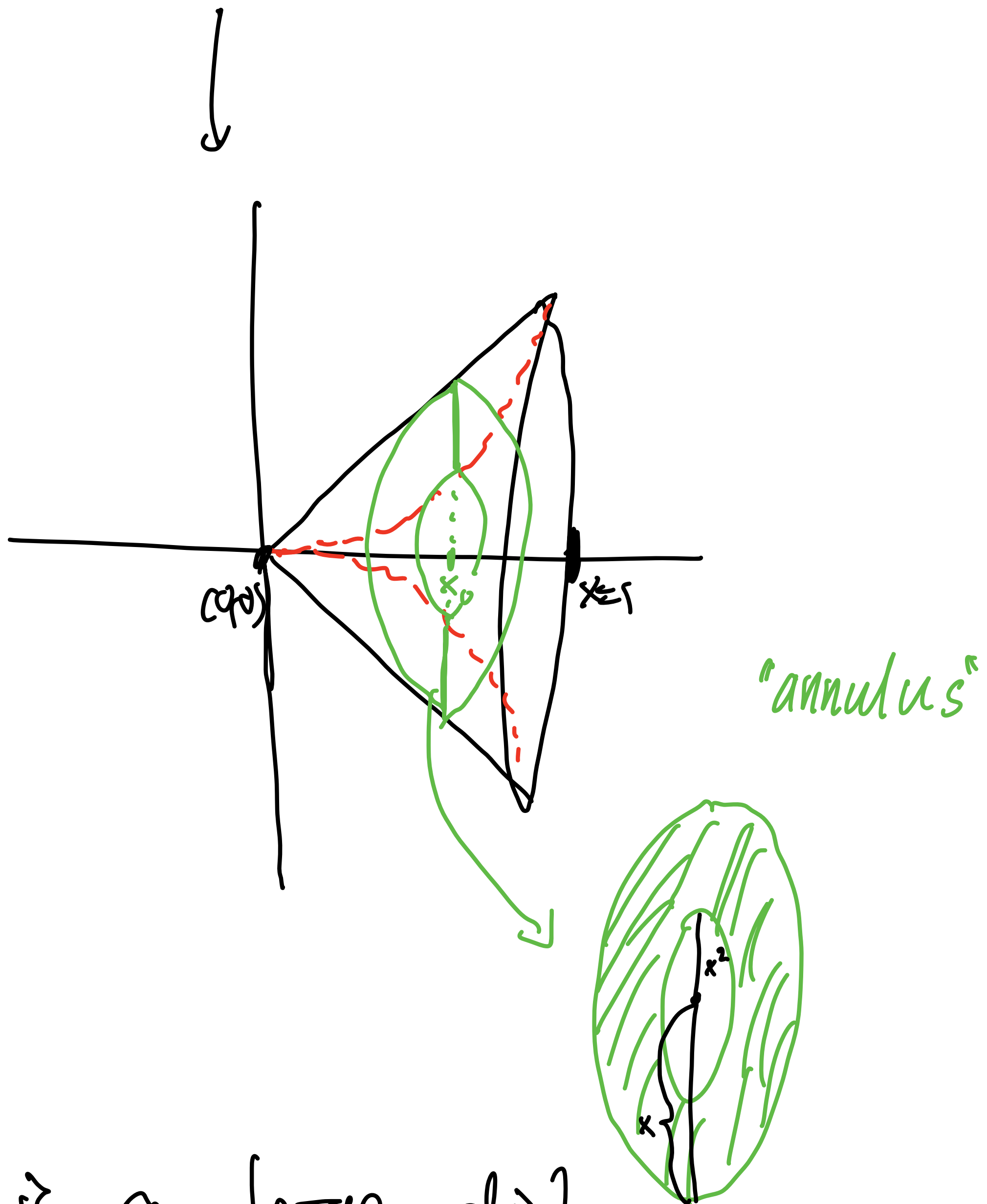
SAME INTEGRAL AS BEFORE!!!!!

A more complicated example:

Ex: Compute the volume of the solid obtained by revolving the bounded region between $y=x$ and $y=x^2$ about the x-axis.

$y=x^2$





- this is a large disk
with a smaller disk removed

so its area is

$$\pi x^2 - \pi (x^2)^2$$

area of the lager circle

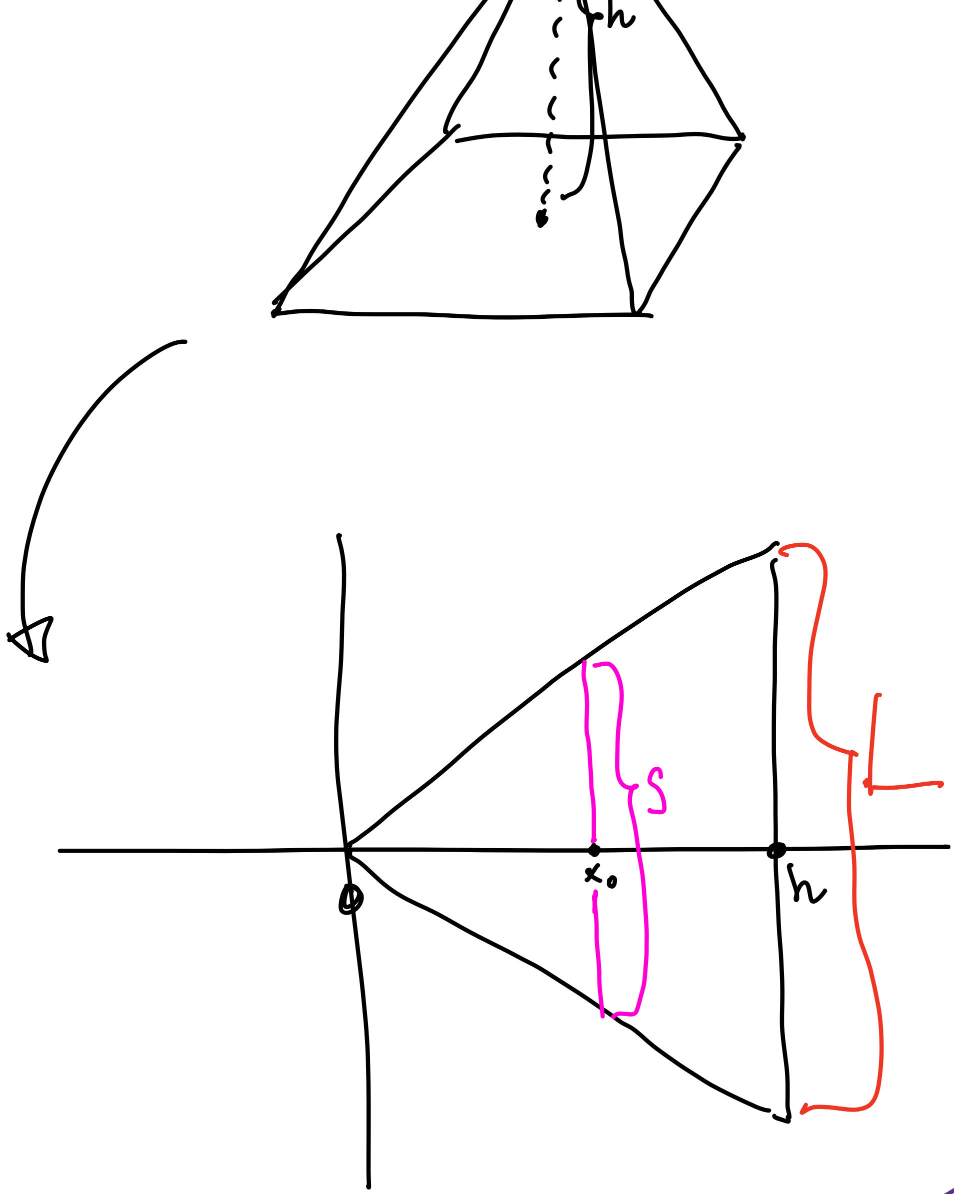
area of the smaller circle

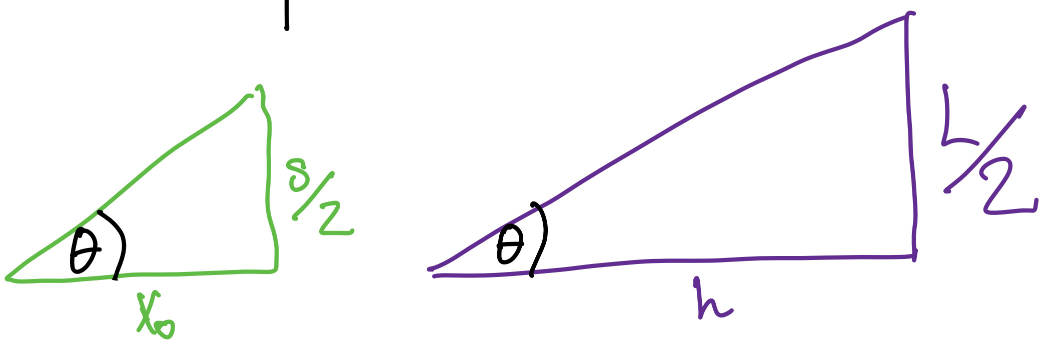
$$= \pi x^2 - \pi x^4$$

Hence the volume of this solid is:

$$\int_0^1 \pi x^2 - \pi x^4 \, dx = \frac{2\pi}{15}$$

<u>Ex:</u> Compute the volume of a pyramid with base a square of side-length L and height h





$$\frac{x_0}{h} = \frac{s/2}{L/2} = \frac{s}{L}$$

$$\Longrightarrow \quad s = \frac{Lx}{h}$$

So the area is $s^2 = \frac{L^2 x^2}{h^2}$

Hence the volume is

$$\int_0^h \frac{L^2}{h^2} x^2 \, dx = \dots = \frac{L^2}{3} \cdot h$$