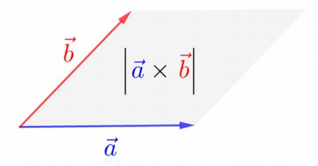
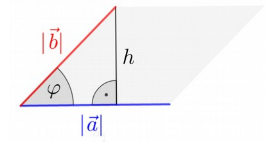
Flächeninhalt eines Parallelogramms per Kreuzprodukt



Formel: $A_{Parallelogramm} = \left|\vec{a}\times\vec{b}\right| = |\vec{n}|$

Herleitung: Die Parallelogrammfläche berechnet sich per „Grundlinie mal Höhe“: $A = |\vec{a}|\cdot h$.
Mit $\sin\varphi = \frac{h}{|\vec{b}|} \;\Leftrightarrow\; h = |\vec{b}|\cdot\sin\varphi$ und $\sin^2\varphi + \cos^2\varphi = 1 \;\Rightarrow\; \sin\varphi = \sqrt{1-\cos^2\varphi}$ [→ *trigonometrischer Pythagoras*]
folgt durch Einsetzen $h = |\vec{b}|\cdot\sqrt{1-\cos^2\varphi}$ und somit:



$$A = |\vec{a}|\cdot|\vec{b}|\cdot\sqrt{1-\cos^2\varphi} = \sqrt{\left(|\vec{a}|\cdot|\vec{b}|\right)^2\cdot(1-\cos^2\varphi)} = \sqrt{\left(|\vec{a}|\cdot|\vec{b}|\right)^2 - \left(|\vec{a}|\cdot|\vec{b}|\right)^2\cdot\cos^2\varphi} = \sqrt{|\vec{a}|^2\cdot|\vec{b}|^2 - \left(|\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi\right)^2}$$

$$= \sqrt{\vec{a}\circ\vec{a}\cdot\vec{b}\circ\vec{b} - \left(\vec{a}\circ\vec{b}\right)^2} = \sqrt{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)(a_1b_1+a_2b_2+a_3b_3)}$$ [*vgl.*→ *Skalarprodukt*]

$$= \sqrt{(\cancel{a_1^2b_1^2} + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + \cancel{a_2^2b_2^2} + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + \cancel{a_3^2b_3^2}) - (\cancel{a_1^2b_1^2} + a_1a_2b_1b_2 + a_1a_3b_1b_3 + a_2a_1b_2b_1 + \cancel{a_2^2b_2^2} + a_2a_3b_2b_3 + a_3a_1b_3b_1 + a_3a_2b_3b_2 + \cancel{a_3^2b_3^2})}$$

$$= \sqrt{a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 - a_1a_2b_1b_2 - a_1a_3b_1b_3 - a_1a_2b_1b_2 - a_2a_3b_2b_3 - a_1a_3b_1b_3 - a_2a_3b_2b_3}$$

$$= \sqrt{(a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2) + (a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2) + (a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2)} = \sqrt{(a_2b_3-a_3b_2)^2 + (a_3b_1-a_1b_3)^2 + (a_1b_2-a_2b_1)^2}$$

$$= \left|\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}\right| = \left|\vec{a}\times\vec{b}\right| = |\vec{n}|\,, \quad \text{qed.}$$