Consider a mass m attached to a compressible spring, with Hooke constant k, as shown in Fig. 1. The motion is horizontal, with an initial displacement of L from the origin, which is at the rest position of the spring. The mass is released from rest, so that the initial velocity is zero. The question is, what is the motion $x(t)$ of the mass, when sliding friction is present? This friction force is not velocity-dependent, and it always acts to oppose the present motion of the mass. We consider three cases, and then some example motions, via plots.

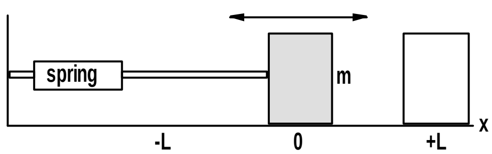

Figure 1. Sketch of spring-mass system.

Case 1: Frictionless

Development

We have the position-dependent force of the spring

$$F = -k\,x$$

as the only force acting on the mass, in the direction of motion, along the x-axis. The mass does not move in any other direction. From Newton's second law,

$$F_{net} = -k\,x = m\,a$$

so that

$$a(x) = -\frac{k}{m}\,x$$

and then

$$\frac{d^2x}{dt^2} = -\alpha\,x\,; \quad \alpha \equiv \frac{k}{m} \tag{1}$$

Solution

This homogeneous second-order ordinary differential equation (ODE) can be solved by several methods; here we use Laplace transforms. Taking transforms of Eq. (1), and using the initial condition of zero velocity and initial position $x(0)=L$ gives

$$s^2\,\Im - s\,L + \alpha\,\Im = 0$$

where s is the Laplace variable and $\Im$ is the Laplace transform. From this we have

$$\Im = \frac{s\,L}{s^2 + \alpha}$$

and from tables of the inverse transform we find that

$$x(t) = \Im^{-1} = L\,\cos\left[t\sqrt{\alpha}\right] \tag{2}$$

The period of this simple harmonic motion is

$$T = \frac{2\pi}{\sqrt{\alpha}} \tag{3}$$

and the mass oscillates between L and $-L$. Qualitatively, if we release the mass from $x=L$, to the right of the zero point, the spring is extended and so it pulls to the left; the mass is accelerated to the left, passing through zero, and it continues to the left. This compresses the spring, which then exerts a force to the right, slowing down the mass, till it stops. Then it is accelerated to the right, passing through zero, and extends the spring out to $x=L$ again, completing one cycle. The alternate extension and compression of the spring provides the force to move the mass. No energy is lost in this frictionless system, so the motion continues indefinitely. Note that the initial position could just as well have been at $-L$, in which case the spring is initially compressed; the motion is still described by Eq. (2).

Maximum Velocity

The velocity of the mass is given by the first time derivative of the position, Eq. (2), so that

$$v(t) = -L\sqrt{\alpha}\,\sin\left[t\sqrt{\alpha}\right] \tag{4}$$

The maximum magnitude of the velocity is attained at a time τ when the sine in Eq. (4) is positive or negative unity, so that we have

$$\tau_1 = \frac{\sin^{-1}(1)}{\sqrt{\alpha}} = \frac{\pi}{2\sqrt{\alpha}}$$

$$\tau_2 = \frac{\sin^{-1}(-1)}{\sqrt{\alpha}} = \frac{3\pi}{2\sqrt{\alpha}}$$

These times correspond to positions

$$x(\tau_1) = L\,\cos\left[\frac{\pi\sqrt{\alpha}}{2\sqrt{\alpha}}\right] = 0$$

$$x(\tau_2) = L\,\cos\left[\frac{3\pi\sqrt{\alpha}}{2\sqrt{\alpha}}\right] = 0$$

and so the velocity is maximized as the mass passes through $x=0$ from either direction.

Maximum Acceleration

The acceleration is the time derivative of Eq. (4), which gives

$$a(t) = -L\alpha\,\cos\left[t\sqrt{\alpha}\right] \tag{5}$$

The magnitude of the acceleration will be maximized when the cosine in Eq. (5) is positive or negative unity, so that

$$\tau_1 = \frac{\cos^{-1}(1)}{\sqrt{\alpha}} = 0$$

$$\tau_2 = \frac{\cos^{-1}(-1)}{\sqrt{\alpha}} = \frac{\pi}{\sqrt{\alpha}}$$

Using these values for time in Eq. (2) to find the corresponding positions, we find that

$$x(\tau_1) = L\cos\left[0\sqrt{\alpha}\right] = L$$

$$x(\tau_2) = L\cos\left[\frac{\pi\sqrt{\alpha}}{\sqrt{\alpha}}\right] = -L$$

and so the acceleration is maximized at the extreme points of the travel. This makes sense, because the spring is at is maximum extension or compression when the mass is at a distance of L from the origin, in either direction.

Case II: Velocity-Dependent "Friction"

In many engineering textbooks[1] this problem is analyzed, using the artifice of "viscous damping" to introduce a tractable form for friction. This sort of energy loss is taken to be proportional to the current velocity of the mass, so that we have, using a similar argument as above on the forces acting on the mass, an ODE of the form

$$\frac{d^2x}{dt^2} = -\alpha x - q\frac{dx}{dt}$$

where q is a damping constant. Solutions of this ODE take the form of exponential-sinusoid products, so that the amplitude of the motion decays away over time. The value of the damping parameter q is central to the form of the solution, which may be under- or over-damped, or "critically" damped; in the latter case there is no oscillation. This form of friction is not especially realistic, and is not what we are after. (Note that, however, this ODE *does* describe a number of applications quite well, notably AC circuits.)

Case III: Sliding Friction

Development

For the spring-mass system, as seen in the real world, we will have the usual kind of sliding friction, where the magnitude of the friction force depends only on the object's mass m, the gravitational acceleration g, and the coefficient of friction μ, so that

$$F_f = \mu F_N = \mu m g$$

The crucial fact about this problem is that *the direction of the friction force is always opposite that of the motion*. This means when the mass is moving from right to left, or the negative sense, the friction force is directed in the positive sense, and vice-versa; also, when the mass is not moving, the friction force vanishes. This seemingly simple situation greatly complicates the solution.

We need to rewrite the force balance, which now becomes

$$F_{net} = F_{spring} \pm F_f = ma = -kx \pm \mu m g$$

where the $\pm$ indicates that this sign can change. Then we have the second-order ODE

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \pm \mu g = -\alpha x \pm \beta$$

The question is, how do we choose the sign for β? Since the friction force always opposes the motion, we can use the sign of the velocity to get the sign of β:

$$\frac{d^2x}{dt^2} = -\alpha x + \beta \quad \text{if } \frac{dx}{dt} < 0$$

$$\frac{d^2x}{dt^2} = -\alpha x - \beta \quad \text{if } \frac{dx}{dt} > 0$$

The first case is motion from right to left, and the second is motion from left to right.

Solution

We can solve these ODEs as before, using Laplace transforms. This time, the ODEs are nonhomogeneous, and we will have, for the same initial displacement $x=L$ and initial velocity of zero,

$$s^2\Im - sL + \alpha\Im = \frac{\beta}{s}$$

This gives the transform

$$\Im = \frac{\beta + s^2 L}{s\left(s^2 + \alpha\right)}$$

the inverse of which is

$$x(t) = \Im^{-1} = \frac{\beta}{\alpha}\left\{1 - \cos\left[t\sqrt{\alpha}\right]\right\} + L\cos\left[t\sqrt{\alpha}\right]$$

Note that this is the sum of the homogeneous solution obtained above (second term), and a particular solution (first term). Rearranging,

$$x(t) = \left(L - \frac{\beta}{\alpha}\right)\cos\left[t\sqrt{\alpha}\right] + \frac{\beta}{\alpha} \qquad (6)$$

which is the form we will use below. If the coefficient of friction is zero, β is zero, and we are back to Eq. (2), as expected. *It is essential to understand that Eq. (6) only applies for the first motion of the mass, from position x=L, while it moves to the left, until it stops*. In the next section we will find when the mass stops, and then consider what happens thereafter.

[1] For example, P.V. O'Neill, *Advanced Engineering Mathematics*, 3rd Ed., Wadsworth (1991), pp. 135-139.

If the initial position is $x(0) = -L$, we need to consider that the direction of the friction force will also be reversed. In this case the initial motion is from left to right, and the ODE solution will yield

$$x(t) = \left(-L + \frac{\beta}{\alpha}\right)\cos\left[t\sqrt{\alpha}\right] - \frac{\beta}{\alpha}$$

Travel Time

To find when the mass will stop, we need the first time derivative of Eq. (6), i.e., the velocity. This will be

$$v(t) = -\left(L - \frac{\beta}{\alpha}\right)\sqrt{\alpha}\,\sin\left[t\sqrt{\alpha}\right]$$

Since this has the same form as Eq. (4), the maximum velocity will again occur as the mass passes through the zero point. In this problem we are interested in finding the time when the mass stops moving, on each pass. The velocity will be zero when the sine is zero, which occurs at time zero (trivial) or at time

$$T = \frac{\sin^{-1}(0)}{\sqrt{\alpha}} = \frac{\pi}{\sqrt{\alpha}}$$

which is half the time we found for the period of the no-friction case. Here, T is the time required for the mass to move from its rightmost to leftmost (or vice-versa) position. This would also be the half-period for the frictionless case, so we see that *the periodicity of the motion is not affected by the presence of friction*.

Rest Positions

For the first slide, from $x=L$ on the right, to some position $x(T)$ on the left where the mass momentarily comes to rest, the time required is T. Then the rest position at the left will be

$$x(T) = \left(L - \frac{\beta}{\alpha}\right)\cos\left[\frac{\pi\sqrt{\alpha}}{\sqrt{\alpha}}\right] + \frac{\beta}{\alpha} = -L + \frac{2\beta}{\alpha}$$

So the leftmost position now is not $-L$, as we had for the frictionless case, but a value somewhat larger (more positive) than that.

Next we consider what happens when the mass moves from this leftmost, rest position $x(T)$ back to the right. Now the sign of β in the ODE is reversed, and we can just flip the signs of the β terms in Eq. (6). However, things are not so simple. We must realize that the ODE solution proceeds from the initial x-value $x(T)$, not from $x=L$, as was used to develop Eq. (6). The initial velocity is still zero, but the initial x is the (rest) position attained when the velocity was zero on the last pass. Using the negative of β (since the mass is moving to the right), and the initial $x(T)$ we just found, the ODE solution now will be

$$x(t) = \left\{\left(-L + \frac{2\beta}{\alpha}\right) + \frac{\beta}{\alpha}\right\}\times$$
$$\cos\left[(t-T)\sqrt{\alpha}\right] - \frac{\beta}{\alpha}, \quad t > T$$

Note that we are using effectively a new time variable, measured from (i.e., zero at) real time $t=T$.

Next we'd like to know when (and thus, where) the mass will stop on this journey to the right. Again we use the velocity, which is

$$v(t) = -\left\{\left(-L + \frac{2\beta}{\alpha}\right) + \frac{\beta}{\alpha}\right\}\sqrt{\alpha}\,\sin\left[(t-T)\sqrt{\alpha}\right]$$

which will be zero when $t = T$, as we already knew, since that was the at-rest starting point for this iteration, and it will also be zero when

$$t - T = \frac{\pi}{\sqrt{\alpha}}$$

Thus we can say that one complete period, i.e., motion from a rest position on the right, to the (momentary) rest position on the left, and back to rest at the right, is $2T$, which is exactly the period for the no-friction case. The rest position at this time $t=2T$ will be

$$x(2T) = \left\{\left(-L + \frac{2\beta}{\alpha}\right) + \frac{\beta}{\alpha}\right\}\cos\left[\frac{\pi\sqrt{\alpha}}{\sqrt{\alpha}}\right] - \frac{\beta}{\alpha}$$
$$= L - \frac{4\beta}{\alpha}$$

We see that the motion to the right does not return to the initial position $x=L$, but rather to a position somewhat less than this.

Piecewise Solution

In both passes done so far, one from right to left, then from left to right, the mass has not had as large an excursion in x as happened for the frictionless case. This is of course what we expect- friction is dissipating the energy of the system, and damping the motion. If we continue the ODE solution process a few more steps we will find a pattern emerging: the rest position at each time T becomes smaller by a decrement

$$\delta = \frac{2\beta}{\alpha}$$

so that each rest position is closer to zero by this amount, compared to the previous rest position's distance from zero. This pattern can be summarized in the following expressions for the x-position of the mass. Note that this solution is piecewise- a different solution applies during each "half-period" or "travel time" T.

First we find the index number n of the transit, for the current time t (the "ceil" function rounds up):

$$n = \text{ceil}\left(\frac{t}{T}\right)$$

Next we define some factors which need to be conditioned on whether the initial position L was positive or negative:

$$k_1 = (-1)^{(n+1)}\left[L - (n-1)\frac{2\beta}{\alpha}\right] + (-1)^n \frac{\beta}{\alpha} \qquad L > 0$$

$$k_2 = (-1)^{(n+1)}\frac{\beta}{\alpha}$$

$$k_1 = (-1)^{(n+1)}\left[L + (n-1)\frac{2\beta}{\alpha}\right] - (-1)^n \frac{\beta}{\alpha} \qquad L < 0$$

$$k_2 = (-1)^n \frac{\beta}{\alpha}$$

With these factors we can write the piecewise solution for the motion:

$$x(n,t) = k_1(n)\cos\left[\left(t - (n-1)T\right)\sqrt{\alpha}\right] + k_2(n) \qquad (7)$$

This can of course be mechanized, to produce plots of the motion, as we will see below.

Endpoint

The final consideration in the analysis of this system is, when will the mass cease to move?

In a frictionless system the motion, once initialized by the input of energy (to displace the mass to some nonzero initial position), will continue "forever." For the friction case just considered, we recognize that the spring force must overcome the friction force at each "rest stop" the mass makes. When the displacement *x(t)* is no longer sufficient to produce a spring force of sufficient magnitude to move the mass against the force of friction, then the mass will not move any further.

In terms of the piecewise solution developed above, we seek N, the maximum number of excursions the mass will experience. When the forces just balance, at a rest position, we have

$$\left|k\,x(NT)\right| = \mu m g$$

so that

$$\left|x(NT)\right| = \frac{\beta}{\alpha}$$

Considering Eq. (7) we see that when k_1 is zero, this condition is met. Thus, for positive L,

$$(-1)^{(N+1)}\left[L - (N-1)\frac{2\beta}{\alpha}\right] = -(-1)^N \frac{\beta}{\alpha}$$

which can be solved for N to give

$$N = \frac{1}{2} + \frac{kL}{2\mu m g} \qquad (8)$$

If the initial displacement is negative, we use the absolute value of L in Eq. (8). Note that if the friction coefficient μ is zero, N is infinite; this means that the motion will not stop. This formulation is based on checking the force balance at each rest position (i.e., exactly zero velocity). However, in practice, the motion can stop when the velocity is "small" and the forces are (nearly) balanced; this need not occur only at a time nT.

As an interesting sidebar, we can estimate the total distance moved by the mass, by considering the potential energy stored in the spring by virtue of its initial displacement L, and recognizing that the friction must dissipate this energy. The spring's potential energy is given by

$$\frac{1}{2}kL^2$$

while the work done by friction is

$$\mu m g D$$

Equating these and solving for D, we must have

$$D = \frac{kL^2}{2\mu m g} = \int_0^{\phi}\left|v(t)\right|\,dt$$

where ϕ is the time when the motion stops. Testing this with the numerical simulation discussed in the next section shows that this relation holds.

Example Solutions

Simulation

A numerical simulation was developed, which solves the ODE for the friction case, using a simple Euler iteration. The time-dependent position, velocity, acceleration, spring force, friction force are found and can be plotted. The program accepts inputs for the spring constant, mass of the object, coefficient of friction, and the initial position. The piecewise solution for the position developed above is also implemented and can be plotted for comparison to the numerical results.

Figure 2 shows pseudocode for the calculations. Note the stopping rule, based on the magnitude of the spring force being less than the friction force, at any time when the magnitude of the velocity is "small" (it need not be exactly zero, i.e., at a time *nT*). The minimum velocity is an input variable. This test can be defeated; the simulation will still produce the slowdown and stopping of the mass, but it will oscillate at the end, as the velocity and acceleration interact to rapidly switch the friction force direction. Physically, it is more sensible to actively terminate the motion, since this is what would happen. The termination test sets the acceleration and velocity exactly to zero, and so the position will stay at the value it had when this test was passed. Of course, at the start of the run, before the loop in Fig. 2 is executed, the velocity v is initialized to zero, and the position x is initialized to L, which is a user input.

```
for t = 0  to  tfinal  step  h
   a = - k * x / m  -  sign(v) * mu * g
   if ( dotest = 1 )
      if ( abs(v) < minvel   &   abs (k*x) < mu*m*g  )
         a = 0
         v = 0
      endif
   endif
   v = v  +  a * h
   x = x  +  v * h
  D = D + abs(v) * h
end loop
```

Figure 2. Pseudocode for simulation main calculations

Sample Runs

In Figure 3 we have an example run, for the parameter values indicated. Time is the horizontal axis, in seconds, and the x-displacement in meters is the vertical axis. The solid line with the symbols is the x-position; the line is the numerical solution and the symbols are from the analytical (piecewise) solution. They agree well. The solid line without symbols is the velocity, and the dotted line is the acceleration. Note the discontinuities in the acceleration, which occur at each rest position, when the friction force suddenly changes direction. These discontinuities are reflected in the velocity curve; the change in slope can just be seen, e.g., at about 8 seconds.

In Fig. 4 we have a shorter run, where the spring force and the initial displacement are smaller. This means that the motion is terminated sooner. The change in slope of the velocity can clearly be seen, at about 3 seconds. This run does not complete one cycle, since it terminates on the negative side of the axis. The analytical solution also has a stopping test, using the derivative of Eq. (7) for the velocity. This velocity is shown in Fig. 4 as the triangles overlaid onto the velocity curve from the numerical solution.

Figure 5 shows the position results for a friction case, with the frictionless solution added (dotted lines). We see that the peaks (rest positions) for the friction solution are occurring at the same time as the peaks of the frictionless system. Thus, the friction is not affecting the frequency of the motion, only the amplitude.

Figure 6 shows the positions and the spring force (solid line) and the friction force (dotted line). The friction force has a constant magnitude, but changes sign at each half-period T, when the velocity is zero.

Figure 7 shows a zoom into the last part of this run, where we see that the magnitude of the spring force has fallen just to the friction force level, and the motion stops when the velocity falls below the threshold level, which in the last segment didn't require the full time T. Thus the estimate N is only approximate, since the last segment may not be completed.

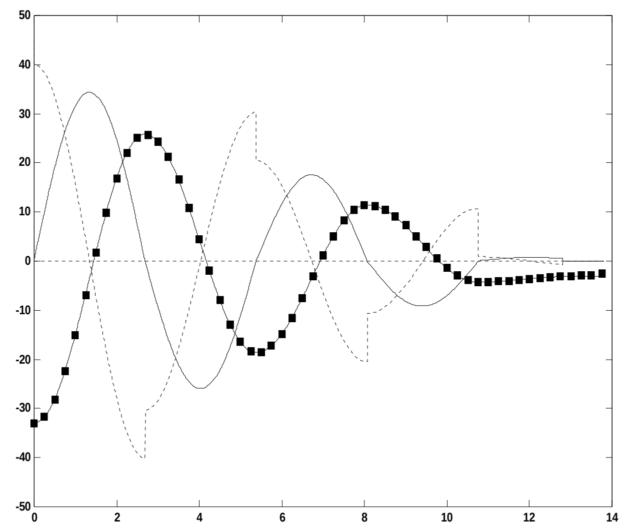

Figure 3. k=30 N/m; μ=0.5; m=22 Kg; L=-33 m.

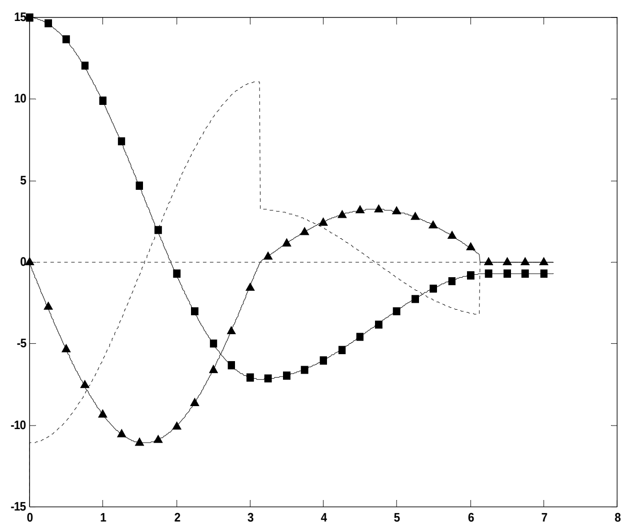

Figure 4. k=15; μ=0.4; m=15; L=15.

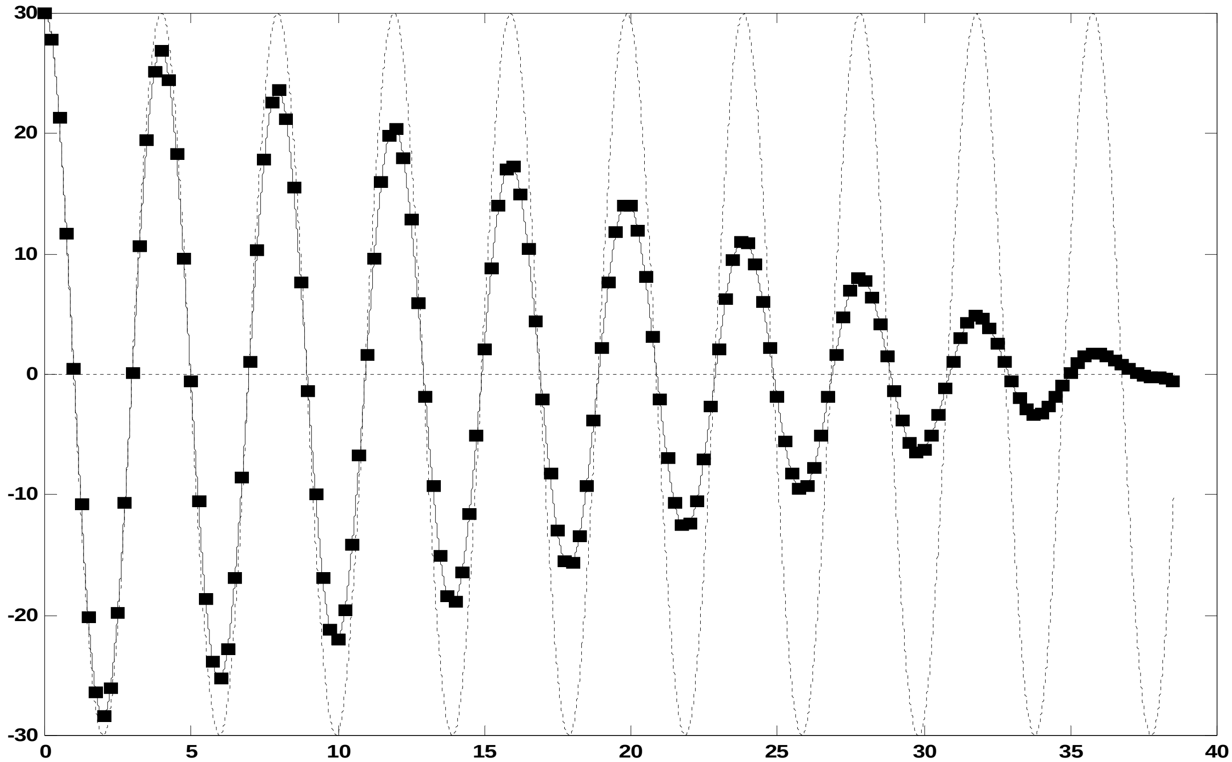

Figure 5. Example run showing frictionless response (dotted line): k=25; μ=0.2; m=10; L=30.

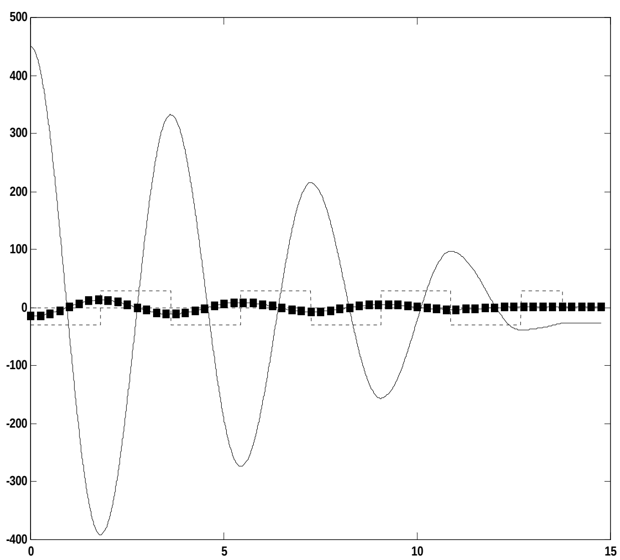

Figure 6. Example run: k=30; μ=0.3; m=10;L=-15. Position and forces.

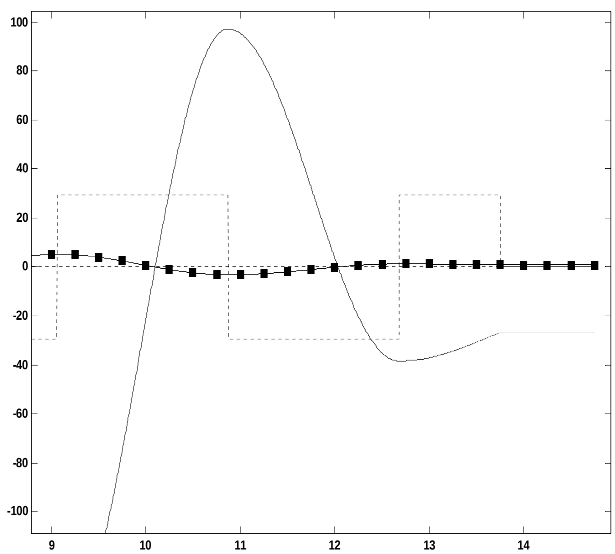

Figure 7. Zoom at end of run in Fig. 6