[MAA 1.11-1.12] COMPLEX NUMBERS (CARTESIAN FORM)

SOLUTIONS

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CARTESIAN FORM

O. Practice questions

1. (a) $\Delta = -16$

(b) $z = \dfrac{8 \pm 4\text{i}}{2} = 4 \pm 2\text{i}$

(c) $(z-4-2\text{i})(z-4-2\text{i}) = (z-4)^2 - 4\text{i}^2 = (z-4)^2 + 4$

2. (a) $\Delta = -144$

$z = \dfrac{8 \pm 12\text{i}}{8} = \dfrac{8}{8} \pm \dfrac{12}{8}\text{i} = 1 \pm \dfrac{3}{2}\text{i}$

(b) (i) $\text{S} = 1 + \dfrac{3}{2}\text{i} + 1 - \dfrac{3}{2}\text{i} = 2$ and $-\dfrac{b}{a} = \dfrac{8}{4} = 2$

(ii) $\text{S} = \left(1 + \dfrac{3}{2}\text{i}\right) + \left(1 - \dfrac{3}{2}\text{i}\right) = 1^2 - \left(\dfrac{3}{2}\text{i}\right)^2 = 1 + \dfrac{9}{4} = \dfrac{13}{4}$ and $\dfrac{c}{a} = \dfrac{13}{4}$

3. (a) (i) $z_1 + z_2 = 13 + 9\text{i}$,

(ii) $z_2 - z_1 = 7 + \text{i}$

(iii) $z_1 z_2 = (3+4\text{i})(10+5\text{i}) = 30 + 15\text{i} + 40\text{i} - 20 = 10 + 55\text{i}$

(iv) $\dfrac{z_2}{z_1} = \dfrac{10+5\text{i}}{3+4\text{i}} = \dfrac{10+5\text{i}}{3+4\text{i}} \times \dfrac{3-4\text{i}}{3-4\text{i}} = \dfrac{50-25\text{i}}{25} = 2 - \text{i}$

(b) (i) $z_1^2 = (3+4\text{i})^2 = 9 + 24\text{i} - 16 = -7 + 24\text{i}$

(ii) $z_1^3 = (3+4\text{i})^3 = 3^3 + 3\times 3^2 \times 4\text{i} + 3 \times 3 \times (4\text{i})^2 + (4\text{i})^3$

$= 27 + 108\text{i} - 144 - 64\text{i} = -117 + 44\text{i}$

(**OR** $(3+4\text{i})^2\,(3+4\text{i}) = (-7+24\text{i})\,(3+4\text{i}) = -117 + 44\text{i}$)

(c) (i) $|z_1| = 5$,

(ii) $|z_2| = \sqrt{125} = 5\sqrt{5}$,

(iii) $|z_2 - z_1| = \sqrt{50} = 5\sqrt{2}$

4. (a) $(1 - \text{i}\sqrt{3})^2 = 1^2 - 2\sqrt{3}\text{i} + \left(\sqrt{3}\text{i}\right)^2 = -2 - 2\sqrt{3}\text{i}$

(b) $(1-\text{i}\sqrt{3})^3 = (1-\text{i}\sqrt{3})^2(1-\text{i}\sqrt{3}) = \left(-2-2\text{i}\sqrt{3}\right)(1-\text{i}\sqrt{3}) = -2 - 6 - 2\text{i}\sqrt{3} + 2\text{i}\sqrt{3} = -8$

5. (a) $a - 2 = 7 \Leftrightarrow a = 9$

$b - 1 = 3 \Leftrightarrow b = 4$

(b) $c - 2 = 0 \Leftrightarrow c = 2$

$d - 1 = 0 \Leftrightarrow d = 1$

6. The final result is 3+i

A. Exam style questions (SHORT)

7. $z = 1 + \frac{i(i+\sqrt{3})}{(i-\sqrt{3})(i+\sqrt{3})} = 1 + \frac{i(i+\sqrt{3})}{-4} = \frac{-5+i\sqrt{3}}{-4} = \frac{5}{4} - \frac{i\sqrt{3}}{4}$

8.

$(1-\sqrt{3}i)(1-\sqrt{3}i) = 1 - 2\sqrt{3}i - 3 \quad (= -2 - 2\sqrt{3}i)$

$(-2-2\sqrt{3}i)(1-\sqrt{3}i) = -8$

$\therefore \frac{1}{(1-\sqrt{3}i)^3} = -\frac{1}{8}$

OR

Attempt at Binomial expansion

$(1-\sqrt{3}i)^3 = 1 + 3(-\sqrt{3}i) + 3(-\sqrt{3}i)^2 + (-\sqrt{3}i)^3$

$= 1 - 3\sqrt{3}i - 9 + 3\sqrt{3}i$

$= -8$

$\therefore \frac{1}{(1-\sqrt{3}i)^3} = -\frac{1}{8}$

9. $z = \frac{2}{(1-i)} \cdot \frac{(1+i)}{(1+i)} + 1 - 4i = 1 + i + 1 - 4i = 2 - 3i$

$z^2 = (2-3i)^2 = -5 - 12i$ (or $x = -5$, $y = -12$)

10. $2(p + iq) = q - ip - 2(1 - i)$

$2p = q - 2$

$2q = -p + 2$

$p = -0.4$, $q = 1.2$

11. $(a+i)(2-bi) = 7 - i \Rightarrow 2a - abi + 2i - bi^2 = 7 - i$

$\Rightarrow 2a - abi + 2i + b = 7 - i$

Equating real and imaginary parts

$2a + b = 7$

$2 - ab = -1 \Rightarrow ab = 3$

Substitution $\Rightarrow 2a^2 - 7a + 3 = 0 \Rightarrow a = 3$ or $a = 1/2$ (rejected).

Therefore, $a = 3$ and $b = 1$

12. $(a + bi)(2 - i) = (5 - i)$

$(a + bi) = \frac{(5-i)}{(2-i)} = \frac{11}{5} + \frac{3}{5}i$ (using a GDC). Therefore $a = \frac{11}{5}$, $b = \frac{3}{5}$

OR $a + bi = \frac{(5-i)}{(2-i)} \times \frac{(2+i)}{(2+i)} = \frac{10 + 5i - 2i + 1}{4 + 1} = \frac{11 + 3i}{5}$ Therefore $a = \frac{11}{5}$, $b = \frac{3}{5}$

OR $(a + bi)(2 - i) = (5 - i)$

$(2a + b) + (2b - a)i = (5 - i)$

$2a + b = 5$

$-a + 2b = -1$ Therefore $a = \frac{11}{5}$, $b = \frac{3}{5}$

13. Let $z = x + \mathrm{i}y$

$(1 - \mathrm{i})(x + \mathrm{i}y) = 1 - 3\mathrm{i}$

$x + y - \mathrm{i}(x - y) = 1 - 3\mathrm{i}$

$$\left.\begin{aligned} x + y &= 1 \\ x - y &= 3 \end{aligned}\right\} \Rightarrow x = 2, y = -1$$

OR

$$(1 - \mathrm{i})z = 1 - 3\mathrm{i} \Leftrightarrow z = \frac{1-3\mathrm{i}}{1-\mathrm{i}} \Leftrightarrow z = \frac{1-3\mathrm{i}}{1-\mathrm{i}} \times \frac{1+\mathrm{i}}{1+\mathrm{i}} \Leftrightarrow z = 2 - \mathrm{i}$$

14. $\mathrm{i}(z + 2) = 1 - 2z \Rightarrow (2 + \mathrm{i})z = 1 - 2\mathrm{i}$

$$\Rightarrow z = \frac{1-2\mathrm{i}}{2+\mathrm{i}} = \frac{1-2\mathrm{i}}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}} = \frac{-5\mathrm{i}}{5} = -\mathrm{i}. \qquad (a = 0, b = -1)$$

15. **METHOD 1**

By rationalizing we obtain

$$\frac{a}{2} + \frac{b}{5} = 3 \text{ and } -\frac{a}{2} + \frac{2b}{5} = 0$$

Solving gives $a = 4, b = 5$.

METHOD 2

$$\frac{a}{1+\mathrm{i}} + \frac{b}{1-2\mathrm{i}} = 3$$

$$a(1-2\mathrm{i}) + b(1+\mathrm{i}) = 3(1-2\mathrm{i})(1+\mathrm{i})$$

$$= 9 - 3\mathrm{i}$$

$\mathrm{Re}(z): a + b = 9$

$\mathrm{Im}(z): -2a + b = -3$

$3a = 12$

$a = 4, b = 5$

16.

$$\frac{z(3-4\mathrm{i})}{(3+4\mathrm{i})(3-4\mathrm{i})} - \frac{5\mathrm{i}(z-1)}{-5\times 5\mathrm{i}\times \mathrm{i}} = \frac{5(3+4\mathrm{i})}{(3-4\mathrm{i})(3+4\mathrm{i})}$$

$$z(3-4\mathrm{i}) - 5\mathrm{i}(z-1) = 15 + 20\mathrm{i}$$

$$z(3-4\mathrm{i}-5\mathrm{i}) = 15 + 20\mathrm{i} - 5\mathrm{i}$$

$$z = \frac{15+15\mathrm{i}}{3-9\mathrm{i}}$$

$$z = -1 + 2\mathrm{i}$$

17. (a) 4+i (b) 3+i

18. Let $z = a + b\mathrm{i}$, so $z^* = a - b\mathrm{i}$

$$|z|^2 = a^2 + b^2 = 20$$

$$\frac{25}{a+b\mathrm{i}} - \frac{15}{a-b\mathrm{i}} = 1 - 8\mathrm{i} \Rightarrow \frac{25(a-b\mathrm{i}) - 15(a+b\mathrm{i})}{a^2+b^2} = 1 - 8\mathrm{i}$$

$$\frac{10a}{20} = 1 \Rightarrow a = 2 \quad \text{and} \quad -\frac{40b}{20} = -8 \Rightarrow b = 4$$

Therefore, $z = 2 + 4\mathrm{i}$

19.

Solving simultaneously

$2z_1 + 3z_2 = 7$

$2z_1 + 2iz_2 = 8 + 8i$

$z_2(2i - 3) = 1 + 8i$

$z_2 = \frac{1+8i}{2i-3} = 1 - 2i$

$z_1 = \frac{7 - 3(1-2i)}{2}$ or $4 + 4i - i(1 - 2i)$

$= 2 + 3i$

20.

METHOD 1

Substituting $z = x + iy$ to obtain $w = \frac{x + yi}{(x + yi)^2 + 1}$

$w = \frac{x + yi}{x^2 - y^2 + 1 + 2xyi}$

Use of $(x^2 - y^2 + 1 - 2xyi)$ to make the denominator real.

$= \frac{(x + yi)(x^2 - y^2 + 1 - 2xyi)}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$

$\text{Im}\, w = \frac{y(x^2 - y^2 + 1) - 2x^2y}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$

$= \frac{y(1 - x^2 - y^2)}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$

$\text{Im}\, w = 0 \quad \Rightarrow 1 - x^2 - y^2 = 0$ *i.e.* $|z| = 1$ as $y \neq 0$

METHOD 2

$w(z^2 + 1) = z$

$w(x^2 - y^2 + 1 + 2ixy) = x + yi$

Equating real and imaginary parts

$w(x^2 - y^2 + 1) = x$ and $2wx = 1,\ y \neq 0$

Substituting $w = \frac{1}{2x}$ to give $\frac{x}{2} - \frac{y^2}{2x} + \frac{1}{2x} = x$

$-\frac{1}{2x}(y^2 - 1) = \frac{x}{2}$ or equivalent

$x^2 + y^2 = 1$, *i.e.* $|z| = 1$ as $y \neq 0$

21. Let $z = x + iy$, $x, y \in \mathbb{R}$.

Then, $|z + 16|^2 = 16|z + 1|^2$

$\Rightarrow (x + 16)^2 + y^2 = 16\{(x + 1)^2 + y^2\}$

$\Rightarrow x^2 + 32x + 256 + y^2 = 16x^2 + 32x + 16 + 16y^2$

$\Rightarrow 15x^2 + 15y^2 = 240$

$\Rightarrow x^2 + y^2 = 16$

Therefore, $|z| = 4$.

POLYNOMIALS

O. Practice questions

22. (a) $f(1)=1-3+7-5=0$

(b) $z=1+2\mathrm{i}\,,\, z=1-2\mathrm{i}$

(c) $(z-1)$, $(z-1-2\mathrm{i})$, $(z-1+2\mathrm{i})$

(d) $f(z)=(z-1)\,(z-1-2\mathrm{i})\,(z-1+2\mathrm{i})=(z-1)\left[(z-1)^2+4\right]=(z-1)\,(z^2-2z+5)$

23. (a) 1+2i, 1–2i, 1, 2

(b) $(z-1-2\mathrm{i})\,(z-1+2\mathrm{i})=(z-1)^2-4\mathrm{i}^2=z^2-2z+5$

(c) $f(z)=2(z^2-3z+2)(z^2-2z+5)=2z^4-10z^3+26z^2-38z+20$

(d) Sum $=1+2+1+2\mathrm{i}+1-2\mathrm{i}=5$, $S=-\dfrac{b}{a}=\dfrac{10}{2}=5$

Product $=1\times2\times(1+2\mathrm{i})\times(1-2\mathrm{i})=2\times(1+4)=10$, $P=\dfrac{e}{a}=\dfrac{20}{2}=10$

24. (a) $f(1)=0\Rightarrow 2+a+26+b+20=0\Rightarrow a+b=-48$

$f(2)=0\Rightarrow 32+8a+104+2b+20=0\Rightarrow 8a+2b=-156\Rightarrow 4a+b=-78$

we find $a=–10$ and $b=–38$

(b) Divide $f(z)$ by $(z-1)(z-2)=z^2-3z+2$ and get $2z^2-4z+10=2(z^2-2z+5)$ with roots 1+2i, 1–2i

25. The two roots 1+2i, 1–2i give the factor z^2-2z+5

Divide by z^2-2z+5 and get z^2-3z+2 with roots 1, 2

26. Factorization finally gives $f(z)=a(z^4-5z^3+13z^2-19z+10)$

But $f(-1)=96\Leftrightarrow 48a=96\Leftrightarrow a=2$

Hence $f(z)=2z^4-10z^3+26z^2-38z+20$

A. Exam style questions (SHORT)

27. If $(z + 2\mathrm{i})$ is a factor then $(z – 2\mathrm{i})$ is also a factor. $(z + 2\mathrm{i})(z – 2\mathrm{i}) = (z^2 + 4)$

The other factor is $(2z^3 – 3z^2 + 8z – 12) \div (z^2 + 4) = (2z – 3)$

The other two factors are $(z – 2\mathrm{i})$ and $(2z–3)$.

28. **METHOD 1**

If $z=-3+2\mathrm{i}$ is a root, then $z=-3-2\mathrm{i}$ is another.

$$P(z)=(z+2)\big(z-(-3+2\mathrm{i})\big)\big(z-(-3-2\mathrm{i})\big)$$

$$=(z+2)\big(z^2+(3+2\mathrm{i})z+(3-2\mathrm{i})z+13\big)$$

$$=(z+2)(z^2+6z+13)\quad =z^3+8z^2+25z+26$$

$a=8\quad b=25\quad c=26$

METHOD 2

$0=-8+4a-2b+c$

$0=9+46\mathrm{i}+a(5-12\mathrm{i})+b(-3+2\mathrm{i})+c$

$-9=5a-3b+c$ and $46=12a-2b$

solving system of three equations $a=8\quad b=25\quad c=26$

29.

METHOD 1

Using factor theorem

Substituting $z=-1-\mathrm{i}$ into $P(z)$

$-(6+n)+(2m-2-n)\mathrm{i}=0$

Equating both real and imaginary parts to zero

Hence $m=-2$ and $n=-6$

METHOD 2

Using Conjugate root theorem

Multiply $(z+1-\mathrm{i})(z+1+\mathrm{i})\quad =z^2+2z+2$

Let $P(z)=(z^2+2z+2)(z-a)$

$-2a=-8\quad a=4$

Hence $m=-2$ and $n=-6$

B. Exam style questions (LONG)

30. (a) $(1+\mathrm{i})^2=1+2\mathrm{i}+\mathrm{i}^2=2\mathrm{i}$

(b) $(1+\mathrm{i})^{4n}$

Let P(n) be the proposition: $(1+\mathrm{i})^{4n}=(-4)^n$

We must first show that P(1) is true.

$(1+\mathrm{i})^4=((1+\mathrm{i})^2)^2=(2\mathrm{i})^2=4(\mathrm{i})^2=(-4)^1$

Next, assume that for some k e $\mathbb{N}^+$

P(k) is true, then show that P(k+1) is true.

P(k): $(1+\mathrm{i})^{4k}=(-4)^k$

Now, $(1+\mathrm{i})^{4(k+1)}=(1+\mathrm{i})^{4k}(1+\mathrm{i})^4$

$=(-4)^k(-4)=(-4)^{k+1}$

Therefore, by mathematical induction P(n) is true for all $n\in\mathbb{N}^+$

(c) $(1+\mathrm{i})^{32}=(1+\mathrm{i})^{4(8)}=(-4)^8\ =65536$

31. theoretical

32. theoretical