A stone is dropped from a tower. At the instant it falls a distance "c" from the top of the tower another stone is dropped from a position which is a distance "b" below the height "h" of the tower. The stones hit the ground at the same time. What is the height of the tower? (Ignore air resistance.) [*Introduction to Dynamics*, Hardenburgh, p45]

SOLUTION

The basic fact we need is that the first stone (S1) will take the same time to hit the ground after it passes through the point *c* (the "trigger" point) as the second stone (S2) needs to fall from height *b* to the ground. We must recognize also that the first stone will have a nonzero initial velocity at point *c* while the second stone has a zero initial velocity ("dropped" not "thrown") when it is released. We take *y* to be positive downward, with zero at the top of the tower (not the ground).

We will need to have the velocity of S1 as it passes through *c*. This can be found from

$$c = \frac{1}{2} g T_c^2 \quad \text{so that} \quad v_c = g T_c = \sqrt{2 c g}$$

where T_c is of course the time needed to fall through the distance *c*. When S1 falls from position *c* to the ground we will have

$$y(\delta) = h = c + v_c \delta + \frac{1}{2} g \delta^2 \tag{1}$$

where δ is the time needed for S1 to hit the ground after passing through *c*.

Now we consider S2. It will need a time τ to fall from its initial position *b* to the ground:

$$h = b + \frac{1}{2} g \tau^2 \quad \text{so that} \quad \tau = \sqrt{\frac{2 (h - b)}{g}}$$

and this is the same time that S1 needs to get from *c* to the ground. Then we can write

$$\frac{1}{2} g \delta^2 + v_c \delta + (c - h) = 0 \quad \text{from which} \quad \delta = \frac{-v_c + \sqrt{v_c^2 - 4 \left(\frac{1}{2} g\right) (c - h)}}{g} = \tau$$

Since the time must be positive we discard the negative root. Now we have, using v_c found above,

$$\frac{-\sqrt{2 c g}}{g} + \frac{1}{g} \sqrt{2 c g - 2 g (c - h)} = \sqrt{\frac{2 (h - b)}{g}}$$

which is

$$\sqrt{2 c g - 2 g (c - h)} - \sqrt{2 g (h - b)} = \sqrt{2 c g}$$

Removing the common factor $\sqrt{2 g}$ we then have

$$\sqrt{h} - \sqrt{h - b} = \sqrt{c} \tag{2}$$

which, with a little algebra (square both sides, collect, square again), finally gives

$$\boxed{h = \frac{(b + c)^2}{4 c}} \tag{3}$$

Next we would like to explore any constraints on the values of the displacements *b* and *c*. There were none given in the problem statement, other than to say that the stones hit the ground at the same time. However, we can easily picture a situation in which this cannot happen. Suppose the "trigger" distance *c* is only a meter above the ground, while the second stone is at a distance *b* only a meter below the top of the tower. Clearly it will be impossible for S2, experiencing free fall, to hit the ground at the same time as S1, which is much closer to the ground, and moving very fast. So, there must be some limitations on the positions *b* and *c* for this scheme to work (i.e., to be able to find the tower height).

Here is one way to investigate this. Let $f = b / c$. Then our result for *h*, above, can be written

$$h = \frac{(f\,c + c)^2}{4\,c} = \frac{c}{4}(1+f)^2$$

But we know that *h* must be greater than *c*, else *c* is below the ground; thus $h / c > 1$ and so

$$\frac{h}{c} = \frac{1}{4}(1+f)^2 \geq 1$$

from which it is easy to find that *f* must be greater than or equal to unity. This in turn means that *b* must be greater than or equal to *c*. That is, ***the drop point of S2 must be below the trigger point of S1***.

Another way to look at the problem is to examine the times needed for S1 and S2 to reach the ground. For S1 we can solve (1) for the time δ, and similarly for S2. If a solution for *h* is to be found, these times must be equal. Let us investigate the conditions under which this can happen.

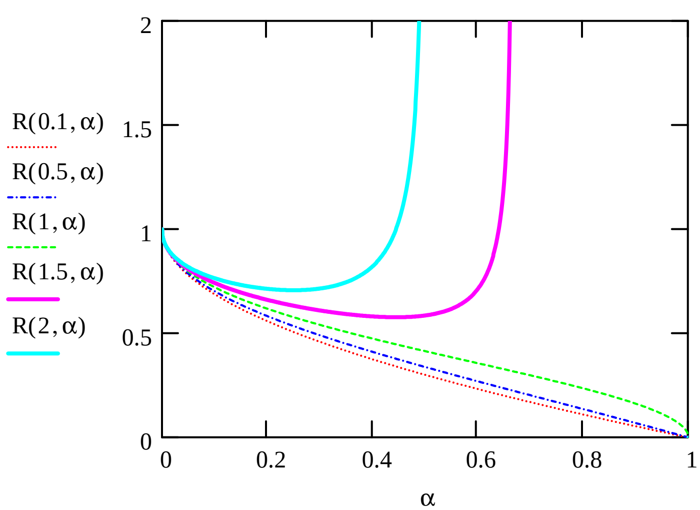
It is easy to show that we can form a ratio of the time for S2 and S1 to hit the ground:

$$R = \frac{\delta}{\tau} = \frac{\sqrt{h} - \sqrt{c}}{\sqrt{h-b}}$$

As before we will use $f = b / c$ so that we can write

$$R(f,\alpha) = \frac{1 - \sqrt{\frac{c}{h}}}{\sqrt{1 - f\left(\frac{c}{h}\right)}} = \frac{1-\sqrt{\alpha}}{\sqrt{1 - f\,\alpha}}$$

where $\alpha = c / h$ and this must be less than or equal to unity. Next we graph this, for several values of *f*



From this plot we see that only values of *f* that are greater than unity will produce a ratio of times that can equal unity. **These are the only times that can produce an estimate for *h***. When *f* is less than or equal to unity, the ratio of times never attains a value of unity, and there can be no solution for *h*.