DERIVATION OF KINEMATIC EQUATIONS

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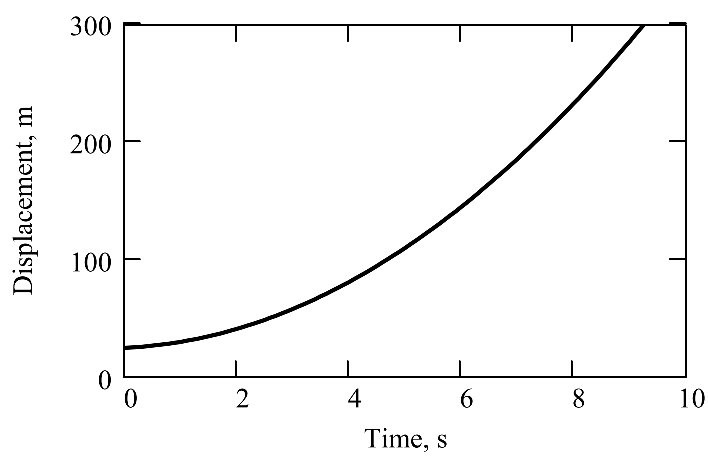
These equations are actually derived using calculus, which was developed in large part, at first, to understand the laws of motion. In calculus two main ideas are essential: (1) the **rate of change** of a function at a point, that is, the slope at that point; (2) the area under a graph across some region of the graph, which relates to a **quantity** of something. This is why we are using these properties of the graphs of displacement, velocity, and acceleration: because the **rate of change** and the **quantity** have physical meanings.

The calculus derivation proceeds from the assumption of a constant acceleration. This leads to a simple second-order ordinary differential equation (ODE). The solution to this ODE first yields an equation for the time-dependent velocity, and then the solution process generates another equation, this one for the time-dependent displacement. These are the two "equations of motion" and deriving them requires that we have the "initial condition" for both the velocity and displacement. The initial conditions are the values of the velocity and displacement at time zero.

It can readily be shown that these equations of motion are the same as we will derive below (Equations 2, 5), without using calculus. From the two equations of motion we can then do some simple algebra to derive several variations, that are useful in solving kinematics problems.

POSITION (DISPLACEMENT) VS. TIME



SLOPE

Since, for accelerated motion, the graph is curved, the slope is not constant. The slope, or rate of change of displacement at any instant is the velocity at that instant. Over an **interval** of time Δt we can find the **average** velocity across that interval. We need to use an average because the velocity is continuously changing during the time interval. We define the average velocity to be the displacement change divided by the time interval:

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

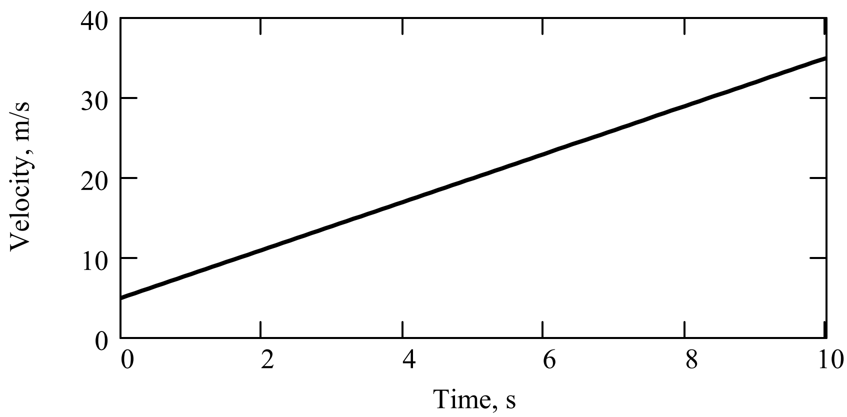
This can also be written as

$$x_{final} = x_{initial} + v_{avg}\,\Delta t \qquad \Delta x := x_{final} - x_{initial} \tag{1}$$

AREA

The area under this graph has no physical interpretation.

VELOCITY VS. TIME



SLOPE

The slope, or rate of change of velocity, of this graph is constant, because we are assuming that the acceleration is constant. The acceleration is the rate of change (slope) of the velocity vs. time graph. So we can write

$$a = \frac{\Delta v}{\Delta t}$$

and this can also be written as

$$v_{final} = v_{initial} + a\,\Delta t \tag{2}$$

AREA

The area under this graph (between the graph and the horizontal axis) across some time interval Δt is the quantity of change in the displacement of the moving object during that time interval. Since the graph is a straight line, we can find this area using the formula from geometry for the area of a trapezoid, or, we can recognize that we have a triangle and a rectangle combined. Then we can write

$$\Delta x = A_{triangle} + A_{rectangle} = \frac{1}{2}\,\Delta t\left(v_{final} - v_{initial}\right) + v_{initial}\,\Delta t$$

and this simplifies to be

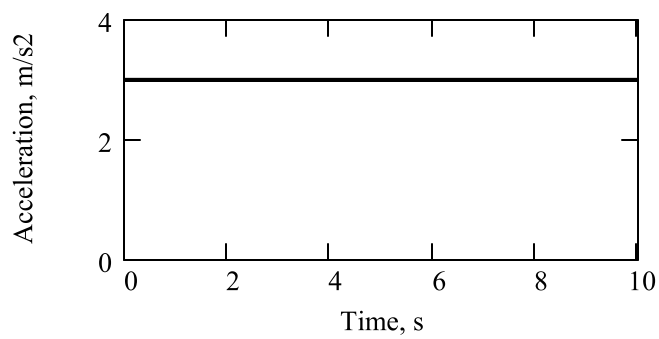
$$\Delta x = \frac{1}{2}\left(v_{initial} + v_{final}\right)\Delta t \tag{3}$$

Next we recognize that we can use this with the definition of the average velocity,

$$\frac{\Delta x}{\Delta t} = v_{avg} = \frac{1}{2}\left(v_{initial} + v_{final}\right) \tag{4}$$

This relation for the average velocity can only be used if the acceleration is constant.

ACCELERATION VS. TIME



SLOPE

Since the acceleration is constant, the slope, or rate of change of acceleration, is zero. (Only with more advanced mathematics can problems with a non-constant acceleration be analyzed.)

AREA

The area under the acceleration graph across some time interval Δt is the quantity of the change in the velocity of the moving object during that time interval. We have already written out the equation for this, above, as equation (2).

DISPLACEMENT VS. TIME EQUATION

One of the most basic and useful kinematics equations tells us the displacement as a function of time. One way to derive this is as follows. We have already found that

$$v_{final} = v_{initial} + a\,\Delta t \quad \text{and} \quad \Delta x = \frac{1}{2}\left(v_{initial} + v_{final}\right)\Delta t$$

These are equations (2) and (3) above. If we substitute (2) into (3) we have

$$\Delta x = \frac{1}{2}\left[v_{initial} + \left(v_{initial} + a\,\Delta t\right)\right]\Delta t$$

which simplifies to

$$\Delta x = v_{initial}\,\Delta t + \frac{1}{2}\,a\,\Delta t^2 \quad \text{or} \quad x_{final} := x_{initial} + v_{initial}\,\Delta t + \frac{1}{2}\,a\,\Delta t^2 \tag{5}$$

Note that the final velocity does not appear in this equation.

VELOCITY / DISPLACEMENT EQUATION

This is a very useful kinematics equation when the time Δt is not given. Using equation (2) we can find that

$$\Delta t = \frac{v_{final} - v_{initial}}{a}$$

and then we can use this in equation (3) to find

$$\Delta x = \frac{1}{2}\left(v_{initial} + v_{final}\right)\left(\frac{v_{final} - v_{initial}}{a}\right) = \frac{v_{final}^2 - v_{initial}^2}{2\ a}$$

which we usually write as

$$v_{final}^2 = v_{initial}^2 + 2\ a\ \Delta x \qquad (6)$$

SUMMARY

Our objective is to be able to solve kinematics problems based on the information given in the problem statement. While in principle we could start from the most basic equations and in effect re-do these derivations, this would waste a lot of time. So we do this algebra once, write down the results, and then choose the proper equation from this set. (See the kinematics "recipe" for a structured way to use these equations; the missing variable is important there, for the process of choosing the appropriate equation.)

	what's missing
$v_{avg} = \frac{1}{2}\left(v_{initial} + v_{final}\right)$	
$x_{final} = x_{initial} + v_{avg}\ \Delta t$	acceleration
$v_{final} = v_{initial} + a\ \Delta t$	displacement Δx
$x_{final} = x_{initial} + v_{initial}\ \Delta t + \frac{1}{2}\ a\ \Delta t^2$	final velocity
$v_{final}^2 = v_{initial}^2 + 2\ a\ \Delta x$	time Δt