Mathematical creativity as a choice

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“Invention is discernment, choice”

“The true work of the inventor consists in choosing among... combinations so as to eliminate the useless ones or rather to avoid the trouble of making them, and the rules which must guide this choice are extremely fine and delicate.”

Poincare, H. (1908). *Science and Method*

Choice in student problem solving

"If the situation is not familiar or there is something non-routine about it, then decision-making is made by a mechanism that can be modeled by…using the subjective expected values of available options, given the orientations of the individual."

Schoenfeld, A. (2010). *How We Think*.

Two stories of creativity and choice

- School project for 9th grade students (15 year old)
- More than 100 students/ 40 projects
- The conduct of the project:
  - Choosing a problem out of about 10 problems
  - Solving the chosen problem
  - Posing/choosing follow-up questions
  - Pursuing the chosen questions
  - Presenting results at the student conference

(Palatnik, 2016, Palatnik & Koichu, 2014, 2015, 2017)

Pizza Problem

Every straight cut divides pizza into two separate pieces. What is the largest number of pieces that can be obtained by *n* straight cuts?

A. Solve for *n* = 1, 2, 3, 4, 5, 6

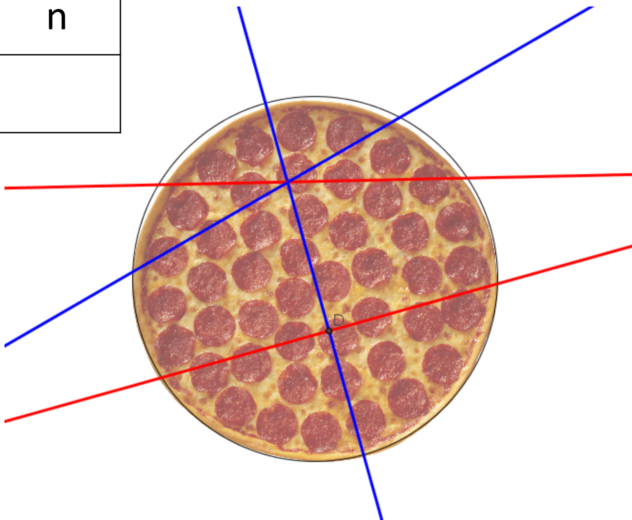
B. Find a recursive formula for the case of n

C. Find an explicit formula

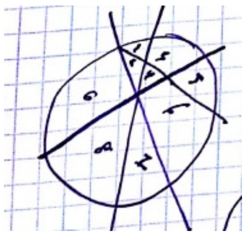
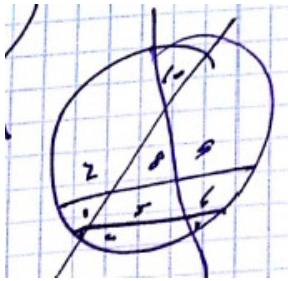
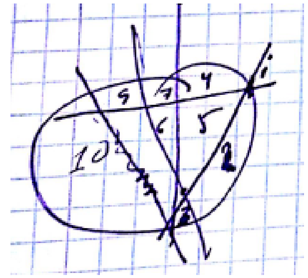
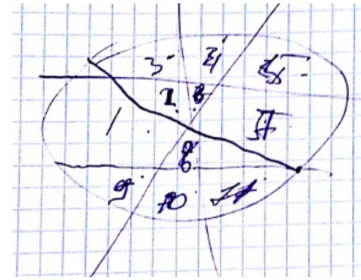
0	1	2	3	4	5	6	7	…	n
1	2	4	7	11	16	22	29	…	

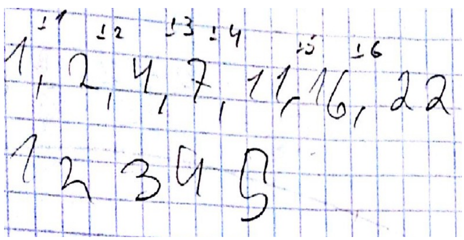
B: $P_n = P_{n-1} + n$, for $n \geq 0$

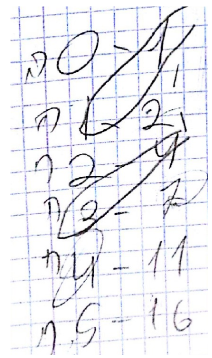
C: $P_n = n(n+1)/2 + 1$



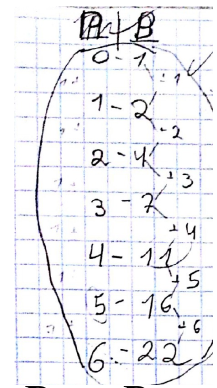
Three-week-long story of Ron in pictures

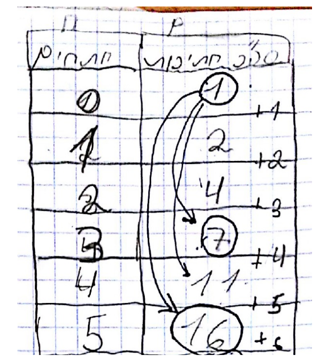




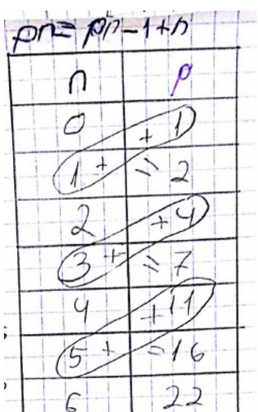
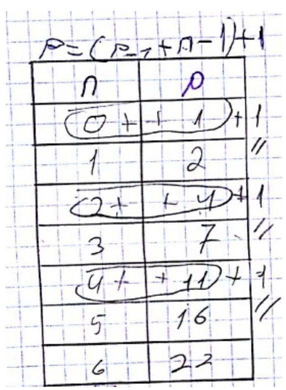
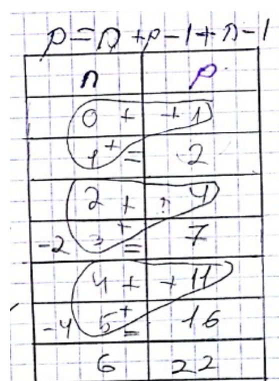
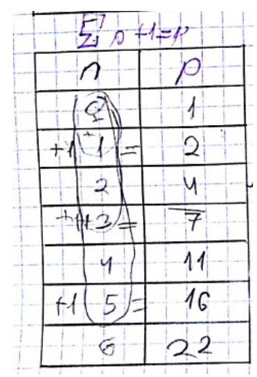
notation



$$P_n = P_{n-1} + n$$



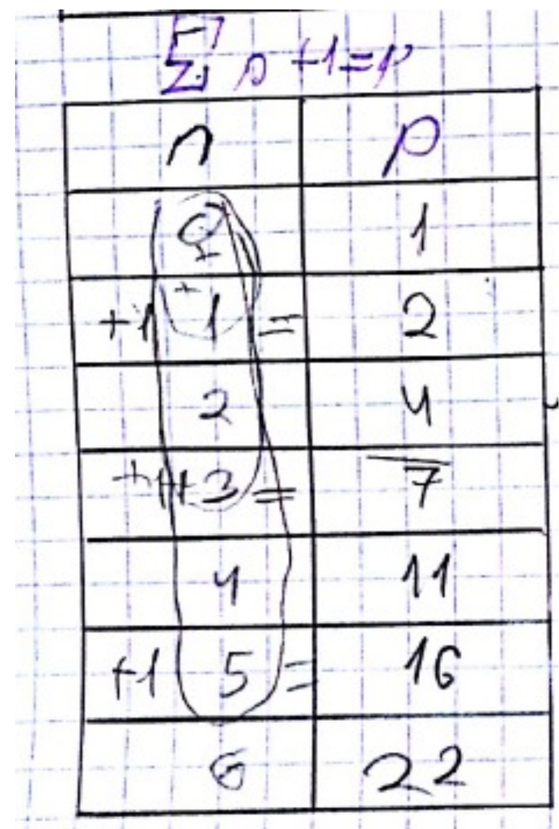
$$P = \sum n + 1$$

$$P = \frac{n(n+1)}{2} + 1$$

Ron about his insight

"I was stuck in one to six. And I just thought…six divided by two gives three. I just thought there's three here, but I could not find the exact connection [to 22]. I do not know why, but I multiplied it by 7, and voila – I got the result."



Ron chooses the follow-up questions

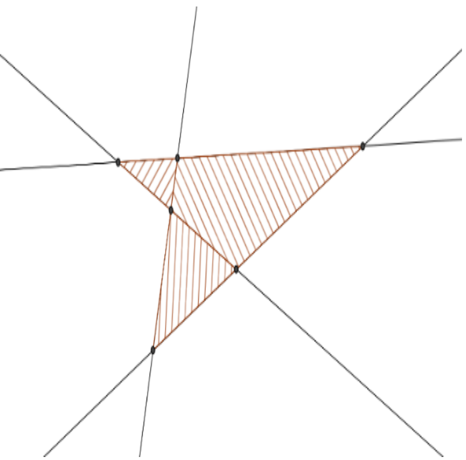
"When we have a formula, but don't know its meaning, it is not interesting... If we'd know how the formula

[($P\downarrow n = n(n+1)/2$ +1)]

is constructed, we would know it for 100%. We got it by chance. So we do not know what it means."

Ron's new goal was two-fold:
"*Why it* [the formula] *works, and prove that it works*"

Four additional weeks of the Ron's story



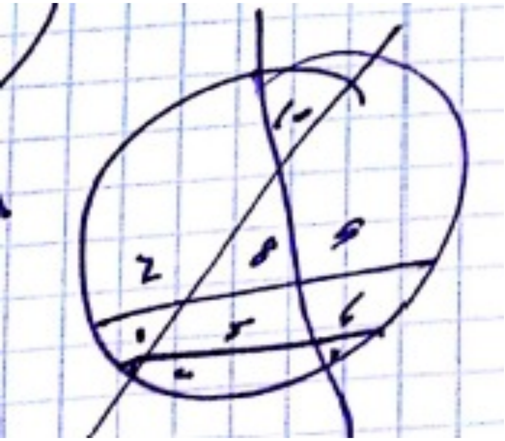
Formula/ expression	What it represents?
$P_n = \frac{n^2+n}{2} + 1$	The sequence representing the maximum numbers of pieces to which a circle pizza can be cut by n cuts
$Z_n = n^2$	The number of mutually exclusive segments on n cutting lines
$X = \frac{n^2+n}{2} + n$	The number of points of intersections between the cutting lines and between the cutting lines and the circle for n cuts
2n	The number of open pieces for n cuts
$\frac{(n-2)^2+n}{2} - 1$	The number of closed pieces for n cuts

Culmination: Ron's big idea

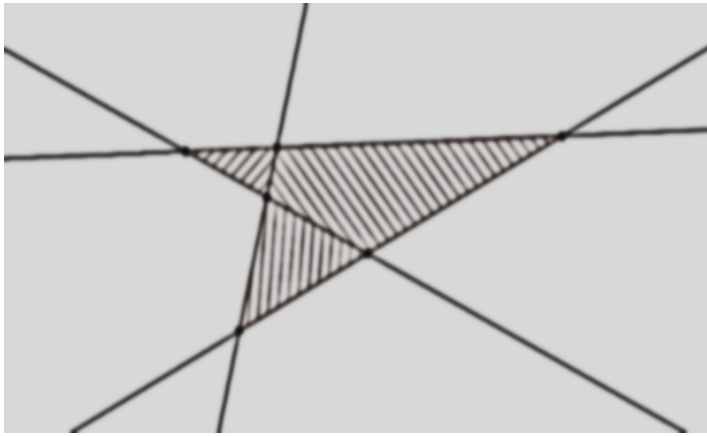
No. of All PIECES = CLOSED PIECES + OPEN PIECES

$$(n-2)\uparrow 2 + n/2 - 1 + 2n = (n-2)\uparrow 2 + n/2 = n(n+1)/2 + 1$$

Ron: We proved it!!



Alik & Boris: What?



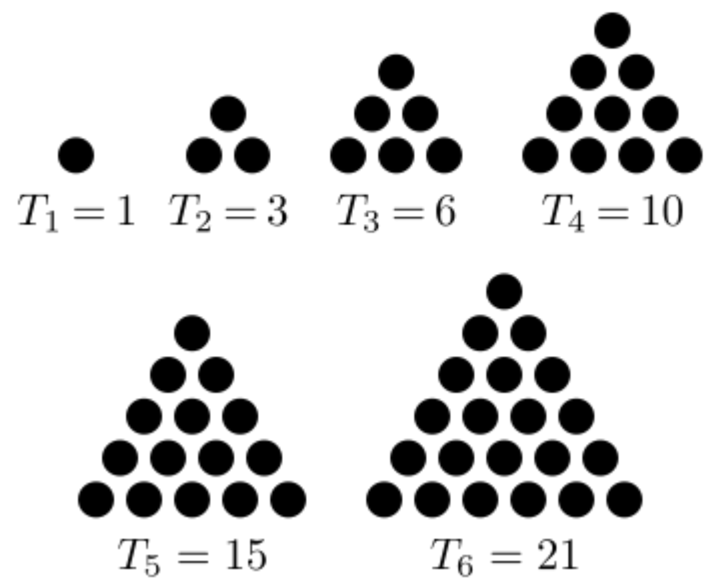
The story of Eli

Two month before the beginning of the project

Eli noted in his essay about the *triangular numbers* :

$T_n = 1+2+3+...+n = n(n+1)/2$

He then wrote: "*A factorial is very similar to a triangular number. Only there is multiplication instead of addition*."



The first week

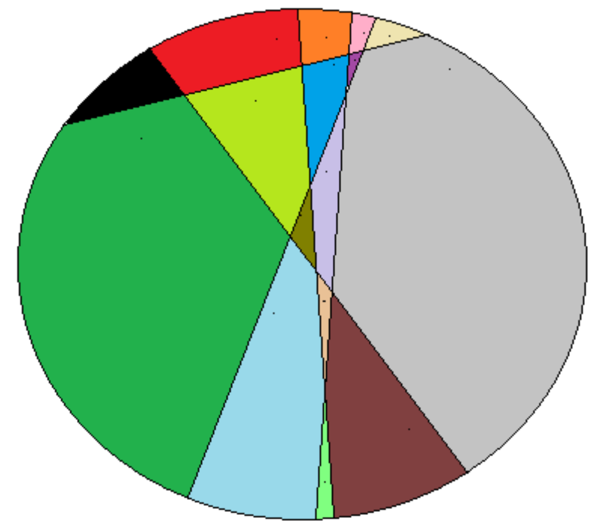
- The student drew dozens of circles representing a pizza and cut them by straight lines.
- “the maximum number of pieces is obtained when each new line intersects all previous lines”.
- The results were recorded in hand-made tables and then in an Excel spreadsheet.
- Eli began exploring the tables for various arithmetic relationships.

Eli's report after the first week

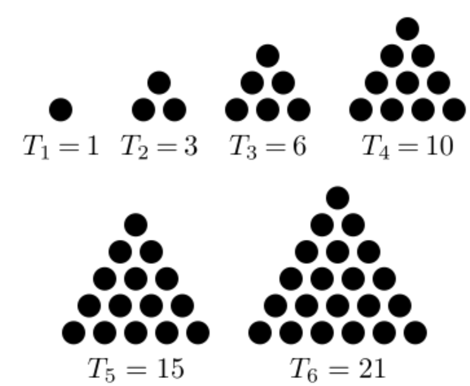
pieces מספר חתיכות	dif. הפרש	cuts מספר חתכים
2	0	1
4	1	2
7	3	3
11	6	4
16	10	5
		6

חיתוך הפיצה

1,3,6,10 הם מספרים משולשיים הנוצרים מההפרש שבין מספר החתיכות של האיבר הקודם לבין מספר החתיכות של האיבר הבא, לבנתיים זו הנוסחא והדרך למציאה של מספר החתכים הכי קטן שיוצר את כמות החתיכות הכי גדולה.



1,3,6,10 are triangular numbers representing the differences between the number of pieces of the previous element and the number of cuts of the next element. For now this is the formula and the way to find the smallest number of cuts that generates the maximum number of pieces.



Three weeks later

Eli sent an email that included the explicit formula for Pizza Problem ($P\downarrow n = n(n+1)/2$ +1) and a detailed explanation.

Briefly, he adjusted the formula

$1+2+3+...+n= n(n+1)/2$, which he knew for triangular number , to the case of the Pizza Problem.

Eli's final e-mail

$$1+2+3+\ldots+n=n(n+1)/2$$

"The factorial of addition is a sum of elements from n to 1. The notation for the factorial of addition, that I propose to be used in calculators, looks like this: $n\underset{+}{!}$, similar to the usual factorial sign, yet instead of a multiplication sign [i.e., dot of the exclamation mark] there is a sign of addition [+] under the exclamation mark."

$$\frac{n^2+n}{2}+1=P_n$$

$$n\underset{+}{!}=\frac{n^2+n}{2}$$

$$n\underset{+}{!}+1=P_n$$

$$n\underset{+}{!}$$

Culmination: Eli’s big idea

Eli proudly presented the solution to Pizza Problem and its connection to triangular numbers, but he was evidently more proud to introduce mathematical notation for "factorial of addition" that he invented, and talked about how useful this sign would be in the future calculators.

$$n\underset{+}{!}$$

What is unusual in Ron and Eli's stories?

- They accepted the initial challenge and demonstrated amazing persistence
- They chose for themselves challenging follow-up questions
- They were creative, though in different ways
- They produced remarkable (for themselves, their classmates and us as teachers and researchers) mathematical results

Choices in Ron and Eli's stories

Ron and Eli could choose:

(1) a problem to solve

(2) a way of dealing with the problem

(3) a direction for a follow-up exploration

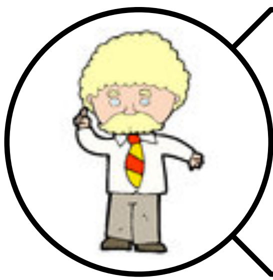
(4) the most important for them (and consequently worthwhile sharing) results of their work

Are there more types of student choices?

Are regular students are in position to choose?

A chain of choices

Student choices: Challenge, approach, mode of participation, extent of collaboration, agent to learn from



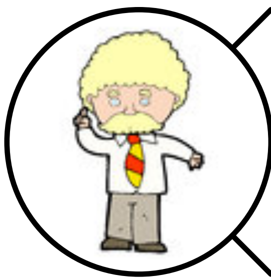
Teacher choices: Task space, declarative knowledge to teach, modes of interactions, expectations…

Commitment to: student, curriculum, mathematics as discipline, society, self…

(Koichu, 2019; in prep.)

Vision: Choice-Affluent Learning Environments

Student choices: Challenge, approach, mode of participation, extent of collaboration, agent to learn from



Teacher choices: Task space, declarative knowledge to teach, modes of interactions, expectations…

Commitment to: student, curriculum, mathematics as discipline, society, self…

(Koichu, 2019; in prep.)

If I had more time…

Research on characteristics of choice-affluent environments

- Koichu, B., & Keller, N. (2019). Creating and sustaining online problem-solving forums: Two perspectives. In P. Liljedahl & L. M. Santos Trigo (Eds.), *Mathematical Problem Solving: ICME 13 Monograph* (pp. 263-287). Springer.
- Koichu, B. (2018). Mathematical problem solving in choice-affluent environments. In Kaiser, G., Forgasz, H, Graven, M., Kuzniak, A., Simmt, E. & Xu, B. (Eds.) *Invited Lectures from the 13th International Congress on Mathematics Education*, ICME-13 Monographs (pp. 307-324). Springer.

Research on teachers' choices

- Koichu, B., & Zazkis, R. (2013). Decoding a proof of Fermat's Little Theorem via script writing. *Journal of Mathematical Behavior, 32*, 364-376.
- Koichu, B., Zaslavsky, O. & Dolev, L. (2015). Effects of variations in task design on mathematics teachers' learning experiences: A case of a sorting task. *Journal of Mathematics Teacher Education*, 19(4), 349-370.

Research on school students' choices

- Koichu, B., Katz, E., & Berman, A. (2017). Stimulating student aesthetic response to mathematical problems by means of manipulating the extent of surprise. *Journal of Mathematical Behavior*, 46, 42-57.
- Palatnik, A., & Koichu, B. (2019). Flashes of creativity. *For the Learning of Mathematics*, 39(2), 8-12.

My favorite questions to myself

- What student choices I tend to support when engaging my students in problem solving?
- What student choices am I aware of?
- What mechanisms are involved in student choices when solving challenging problems?
- How can I support the desirable choices without choosing for the students?
- **How does technology (e.g., GeoGebra) influence students' choices and creativity?**

Thanks for choosing to be here!

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