[MAA 3.6]

TRIGONOMETRIC EQUATIONS

SOLUTIONS

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O. Practice questions

1.

$\sin^{-1}(1) =$	$90°$
$\sin^{-1}\left(\frac{1}{2}\right) =$	$30°$
$\sin^{-1}(0) =$	$0°$
$\sin^{-1}\left(-\frac{1}{2}\right) =$	$-30°$
$\sin^{-1}(-1) =$	$-90°$

$\cos^{-1}(1) =$	$0°$
$\cos^{-1}\left(\frac{1}{2}\right) =$	$60°$
$\cos^{-1}(0) =$	$90°$
$\cos^{-1}\left(-\frac{1}{2}\right) =$	$120°$
$\sin^{-1}(-1) =$	$180°$

2.

	in degrees	*in radians*
$\sin x = \frac{1}{2}$	$x = 30^\circ + 360^\circ k$, $x = 150^\circ + 360^\circ k$	$x = \frac{\pi}{6} + 2k\pi$, $x = \frac{5\pi}{6} + 2k\pi$
$\sin x = \frac{\sqrt{2}}{2}$	$x = 45^\circ + 360^\circ k$, $x = 135^\circ + 360^\circ k$	$x = \frac{\pi}{4} + 2k\pi$, $x = \frac{3\pi}{4} + 2k\pi$
$\sin x = \frac{\sqrt{3}}{2}$	$x = 60^\circ + 360^\circ k$, $x = 120^\circ + 360^\circ k$	$x = \frac{\pi}{3} + 2k\pi$, $x = \frac{2\pi}{3} + 2k\pi$
$\sin x = -\frac{1}{2}$	$x = -30^\circ + 360^\circ k$, $x = -150^\circ + 360^\circ k$	$x = -\frac{\pi}{6} + 2k\pi$, $x = -\frac{5\pi}{6} + 2k\pi$
$\sin x = -\frac{\sqrt{2}}{2}$	$x = -45^\circ + 360^\circ k$, $x = -135^\circ + 360^\circ k$	$x = -\frac{\pi}{4} + 2k\pi$, $x = -\frac{3\pi}{4} + 2k\pi$
$\sin x = -\frac{\sqrt{3}}{2}$	$x = -60^\circ + 360^\circ k$, $x = -120^\circ + 360^\circ k$	$x = -\frac{\pi}{3} + 2k\pi$, $x = -\frac{2\pi}{3} + 2k\pi$

$\sin x = 0$	$x = 180^\circ k$	$x = k\pi$
$\sin x = 1$	$x = 90^\circ + 360^\circ k$	$x = \frac{\pi}{2} + 2k\pi$
$\sin x = -1$	$x = -90^\circ + 360^\circ k$	$x = -\frac{\pi}{2} + 2k\pi$

Notice: For $\sin x = 0$ you could also use **two** formulas instead of **one**, as follows

$\sin x = 0$	$x = 360^\circ k$ $x = 180^\circ + 360^\circ k$	$x = 2k\pi$ $x = \pi + 2k\pi$

3.

	in degrees	*in radians*
$\cos x = \frac{1}{2}$	$x = 60^\circ + 360^\circ k$, $x = -60^\circ + 360^\circ k$	$x = \frac{\pi}{3} + 2k\pi$, $x = -\frac{\pi}{3} + 2k\pi$
$\cos x = \frac{\sqrt{2}}{2}$	$x = 45^\circ + 360^\circ k$, $x = -45^\circ + 360^\circ k$	$x = \frac{\pi}{4} + 2k\pi$, $x = -\frac{\pi}{4} + 2k\pi$
$\cos x = \frac{\sqrt{3}}{2}$	$x = 30^\circ + 360^\circ k$, $x = -30^\circ + 360^\circ k$	$x = \frac{\pi}{6} + 2k\pi$, $x = -\frac{\pi}{6} + 2k\pi$
$\cos x = -\frac{1}{2}$	$x = 120^\circ + 360^\circ k$, $x = -120^\circ + 360^\circ k$	$x = \frac{2\pi}{3} + 2k\pi$, $x = -\frac{2\pi}{3} + 2k\pi$
$\cos x = -\frac{\sqrt{2}}{2}$	$x = 135^\circ + 360^\circ k$, $x = -135^\circ + 360^\circ k$	$x = \frac{3\pi}{4} + 2k\pi$, $x = -\frac{3\pi}{4} + 2k\pi$
$\cos x = -\frac{\sqrt{3}}{2}$	$x = 150^\circ + 360^\circ k$, $x = -150^\circ + 360^\circ k$	$x = \frac{5\pi}{6} + 2k\pi$, $x = -\frac{5\pi}{6} + 2k\pi$
$\cos x = 0$	$x = 90^\circ + 180^\circ k$	$x = \frac{\pi}{2} + k\pi$
$\cos x = 1$	$x = 360^\circ k$	$x = 2k\pi$
$\cos x = -1$	$x = 180^\circ + 360^\circ k$	$x = \pi + 2k\pi$

Notice: For $\cos x = 0$ you could also use **two** formulas instead of **one**, as follows

$\cos x = 0$	$x = 90^\circ k$ $x = -90^\circ k$	$x = \pi/2 + 2k\pi$ $x = -\pi/2 + 2k\pi$

4.

$\tan^{-1}\left(\sqrt{3}\right) =$	$60°$
$\tan^{-1}(1) =$	$45°$
$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) =$	$30°$
$\tan^{-1} 0 =$	$0°$

$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) =$	$-30°$
$\tan^{-1}(-1) =$	$-45°$
$\tan^{-1}\left(-\sqrt{3}\right) =$	$-60°$

5.

	in degrees	*in radians*
$\tan x = \sqrt{3}$	$x = 60^\circ + 180^\circ k$	$x = \frac{\pi}{3} + k\pi$
$\tan x = 1$	$x = 45^\circ + 180^\circ k$	$x = \frac{\pi}{4} + k\pi$
$\tan x = \frac{\sqrt{3}}{3}$	$x = 30^\circ + 180^\circ k$	$x = \frac{\pi}{6} + k\pi$
$\tan x = 0$	$x = 180^\circ k$	$x = k\pi$
$\tan x = -\frac{\sqrt{3}}{3}$	$x = -30^\circ + 180^\circ k$	$x = -\frac{\pi}{6} + k\pi$
$\tan x = -1$	$x = -45^\circ + 180^\circ k$	$x = -\frac{\pi}{4} + k\pi$
$\tan x = -\sqrt{3}$	$x = -60^\circ + 180^\circ k$	$x = -\frac{\pi}{3} + k\pi$

6.

	in degrees $-180^\circ \le x \le 180^\circ$	*in radians* $-\pi \le x \le \pi$
$\sin x = \frac{1}{2}$	$x = 30^\circ, \quad x = 150^\circ$	$x = \frac{\pi}{6}, \quad x = \frac{5\pi}{6}$
$\sin x = \frac{\sqrt{2}}{2}$	$x = 45^\circ, \quad x = 135^\circ$	$x = \frac{\pi}{4}, \quad x = \frac{3\pi}{4}$
$\sin x = \frac{\sqrt{3}}{2}$	$x = 60^\circ, \quad x = 120^\circ$	$x = \frac{\pi}{3}, \quad x = \frac{2\pi}{3}$
$\sin x = -\frac{1}{2}$	$x = -30^\circ, \quad x = -150^\circ$	$x = -\frac{\pi}{6}, \quad x = -\frac{5\pi}{6}$
$\sin x = -\frac{\sqrt{2}}{2}$	$x = -45^\circ, \quad x = -135^\circ$	$x = -\frac{\pi}{4}, \quad x = -\frac{3\pi}{4}$
$\sin x = -\frac{\sqrt{3}}{2}$	$x = -60^\circ, \quad x = -120^\circ$	$x = -\frac{\pi}{3}, \quad x = -\frac{2\pi}{3}$

7.

	in degrees $-180^\circ \le x \le 180^\circ$	*in radians* $-\pi \le x \le \pi$
$\cos x = \frac{1}{2}$	$x = 60^\circ$, $x = -60^\circ$	$x = \frac{\pi}{3}$, $x = -\frac{\pi}{3}$
$\cos x = \frac{\sqrt{2}}{2}$	$x = 45^\circ$, $x = -45^\circ$	$x = \frac{\pi}{4}$, $x = -\frac{\pi}{4}$
$\cos x = \frac{\sqrt{3}}{2}$	$x = 30^\circ$, $x = -30^\circ$	$x = \frac{\pi}{6}$, $x = -\frac{\pi}{6}$
$\cos x = -\frac{1}{2}$	$x = 120^\circ$, $x = -120^\circ$	$x = \frac{2\pi}{3}$, $x = -\frac{2\pi}{3}$
$\cos x = -\frac{\sqrt{2}}{2}$	$x = 135^\circ$, $x = -135^\circ$	$x = \frac{3\pi}{4}$, $x = -\frac{3\pi}{4}$
$\cos x = -\frac{\sqrt{3}}{2}$	$x = 150^\circ$, $x = -150^\circ$	$x = \frac{5\pi}{6}$, $x = -\frac{5\pi}{6}$

8.

	in degrees $0^\circ \le x \le 360^\circ$	*in radians* $0 \le x \le 2\pi$
$\sin x = 0$	$x = -180^\circ, x = 0^\circ, x = 180^\circ$	$x = 0, x = \pi, x = 2\pi$
$\sin x = 1$	$x = 90^\circ$	$x = \pi/2$
$\sin x = -1$	$x = 270^\circ$	$x = 3\pi/2$
$\cos x = 0$	$x = 90^\circ$, $x = -90^\circ$	$x = \pi/2$, $x = 3\pi/2$
$\cos x = 1$	$x = 0^\circ, x = 360^\circ$	$x = 0$, $x = 2\pi$
$\cos x = -1$	$x = 180^\circ$	$x = \pi$,

9.

	in degrees $-180^\circ \le x \le 180^\circ$	*in radians* $-\pi \le x \le \pi$
$\sin x = 0$	$x = -180^\circ, x = 0^\circ, x = 180^\circ$	$x = -\pi, x = 0, x = \pi$
$\sin x = 1$	$x = 90^\circ$	$x = \pi/2$
$\sin x = -1$	$x = -90^\circ$	$x = -\pi/2$
$\cos x = 0$	$x = 90^\circ$, $x = -90^\circ$	$x = \pi/2$, $x = -\pi/2$
$\cos x = 1$	$x = 0^\circ$	$x = 0$
$\cos x = -1$	$x = 180^\circ$, $x = -180^\circ$	$x = \pi$, $x = -\pi$

10. (a) $\sin x = \frac{1}{2}$

$x = 30^\circ + 360^\circ k$: $x = 30^\circ,\ x = 390^\circ$

$x = 150^\circ + 360^\circ k$: $x = 150^\circ,\ x = 510^\circ$

(b) $\sin 2x = \frac{1}{2}$

$2x = 30^\circ + 360^\circ k \Rightarrow x = 15^\circ + 180^\circ k$: $x = 15^\circ,\ x = 195^\circ$

$2x = 150^\circ + 360^\circ k \Rightarrow x = 75^\circ + 180^\circ k$: $x = 75^\circ,\ x = 255^\circ$

(c) $\sin 3x = \frac{1}{2}$

$3x = 30^\circ + 360^\circ k \Rightarrow x = 10^\circ + 120^\circ k$: $x = 10^\circ,\ x = 130^\circ,\ x = 250^\circ$

$3x = 150^\circ + 360^\circ k \Rightarrow x = 50^\circ + 120^\circ k$: $x = 50^\circ,\ x = 170^\circ,\ x = 290^\circ$

11. (a) $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6} + 2k\pi = \frac{\pi + 12k\pi}{6}$: $x = \frac{\pi}{6},\ x = \frac{13\pi}{6}$

$x = \frac{5\pi}{6} + 2k\pi = \frac{5\pi + 12k\pi}{6}$: $x = \frac{5\pi}{6},\ x = \frac{17\pi}{6}$

(b) $\sin 2x = \frac{1}{2}$

$2x = \frac{\pi}{6} + 2k\pi \Rightarrow x = \frac{\pi + 12k\pi}{12}$: $x = \frac{\pi}{12},\ x = \frac{13\pi}{12}$

$2x = \frac{5\pi}{6} + 2k\pi \Rightarrow x = \frac{5\pi + 12k\pi}{12}$: $x = \frac{5\pi}{12},\ x = \frac{17\pi}{12}$

(c) $\sin 3x = \frac{1}{2}$

$3x = \frac{\pi}{6} + 2k\pi \Rightarrow x = \frac{\pi + 12k\pi}{18}$: $x = \frac{\pi}{18},\ x = \frac{13\pi}{18},\ x = \frac{25\pi}{18}$

$3x = \frac{5\pi}{6} + 2k\pi \Rightarrow x = \frac{5\pi + 12k\pi}{18}$: $x = \frac{5\pi}{18},\ x = \frac{17\pi}{18},\ x = \frac{29\pi}{18}$

12. (a) $\cos x = \frac{1}{2}$

$x = 60° + 360°k$: $x = 60°,\ x = 420°$

$x = -60° + 360°k$: $x = 300°,\ x = 660°$

(b) $\cos 2x = \frac{1}{2}$

$2x = 60° + 360°k \Rightarrow x = 30° + 180°k$: $x = 30°,\ x = 210°$

$2x = -60° + 360°k \Rightarrow x = -30° + 180°k$: $x = 150°,\ x = 330°$

(c) $\cos 3x = \frac{1}{2}$

$3x = 60° + 360°k \Rightarrow x = 20° + 120°k$: $x = 20°,\ x = 140°,\ x = 260°$

$3x = -60° + 360°k \Rightarrow x = -20° + 120°k$: $x = 100°,\ x = 220°,\ x = 340°$

13. (a) $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3} + 2k\pi = \frac{\pi + 6k\pi}{3}$: $x = \frac{\pi}{3},\ x = \frac{7\pi}{3}$

$x = -\frac{\pi}{3} + 2k\pi = \frac{-\pi + 6k\pi}{3}$: $x = \frac{5\pi}{3},\ x = \frac{11\pi}{3}$

(b) $\cos 2x = \frac{1}{2}$

$2x = \frac{\pi}{3} + 2k\pi \Rightarrow x = \frac{\pi + 6k\pi}{6}$: $x = \frac{\pi}{6},\ x = \frac{7\pi}{6}$

$2x = -\frac{\pi}{3} + 2k\pi \Rightarrow x = \frac{-\pi + 6k\pi}{6}$: $x = \frac{5\pi}{6},\ x = \frac{11\pi}{6}$

(c) $\cos 3x = \frac{1}{2}$

$3x = \frac{\pi}{3} + 2k\pi \Rightarrow x = \frac{\pi + 6k\pi}{9}$: $x = \frac{\pi}{9},\ x = \frac{7\pi}{9},\ x = \frac{13\pi}{9}$

$3x = -\frac{\pi}{3} + 2k\pi \Rightarrow x = \frac{-\pi + 6k\pi}{9}$: $x = \frac{5\pi}{9},\ x = \frac{11\pi}{9},\ x = \frac{17\pi}{9}$

14.

	in degrees $0^\circ \le x \le 360^\circ$	*in radians* $0 \le x \le 2\pi$
$\tan x = \sqrt{3}$	$x = 60^\circ$, $x = 240^\circ$	$x = \pi/3$, $x = 4\pi/3$
$\tan x = 1$	$x = 45^\circ$, $x = 225^\circ$	$x = \pi/4$, $x = 5\pi/4$
$\tan x = \sqrt{3}/3$	$x = 30^\circ$, $x = 210^\circ$	$x = \pi/6$, $x = 7\pi/6$
$\tan x = 0$	$x = 0^\circ$, $x = 180^\circ$, $x = 360^\circ$	$x = 0$, $x = \pi$, $x = 2\pi$
$\tan x = -\sqrt{3}/3$	$x = 150^\circ$, $x = 330^\circ$	$x = 5\pi/6$, $x = 11\pi/6$
$\tan x = -1$	$x = 135^\circ$, $x = 315^\circ$	$x = 3\pi/4$, $x = 7\pi/4$
$\tan x = -\sqrt{3}$	$x = 120^\circ$, $x = 300^\circ$	$x = 2\pi/3$, $x = 5\pi/3$

15.

	in degrees $-180^\circ \le x \le 180^\circ$	*in radians* $-\pi \le x \le \pi$
$\tan x = \sqrt{3}$	$x = 60^\circ$, $x = -120^\circ$	$x = \pi/3$, $x = -2\pi/3$
$\tan x = 1$	$x = 45^\circ$, $x = -135^\circ$	$x = \pi/4$, $x = -3\pi/4$
$\tan x = \sqrt{3}/3$	$x = 30^\circ$, $x = -150^\circ$	$x = \pi/6$, $x = -5\pi/6$
$\tan x = 0$	$x = 0^\circ$, $x = 180^\circ$, $x = -180^\circ$	$x = 0$, $x = \pi$, $x = -\pi$
$\tan x = -\sqrt{3}/3$	$x = -30^\circ$, $x = 150^\circ$	$x = -\pi/6$, $x = 5\pi/6$
$\tan x = -1$	$x = -45^\circ$, $x = 135^\circ$	$x = -\pi/4$, $x = 3\pi/4$
$\tan x = -\sqrt{3}$	$x = -60^\circ$, $x = 120^\circ$	$x = -\pi/3$, $x = 2\pi/3$

16. (a) $\tan 2x = 0$

$2x = 180^\circ k \Rightarrow x = 90^\circ k$: $x = 0^\circ,\ x = 90^\circ,\ x = 180^\circ.$

(b) $\tan 3x = 0$

$3x = 180^\circ k \Rightarrow x = 60^\circ k$: $x = 0^\circ,\ x = 60^\circ,\ x = -60^\circ.$

(c) $\tan 3x = 1$

$3x = 45^\circ + 180^\circ k \Rightarrow x = 15^\circ + 60^\circ k$: $x = 15^\circ,\ x = 75^\circ,\ x = -45^\circ.$

(d) $\tan 3x = -1$

$3x = -45^\circ + 180^\circ k \Rightarrow x = -15^\circ + 60^\circ k$: $x = -15^\circ,\ x = 45^\circ,\ x = -75^\circ.$

17. (a) $\tan 2x = 0$

$2x = k\pi \Rightarrow x = \dfrac{k\pi}{2}$: $x = 0,\ x = \dfrac{\pi}{2},\ x = \pi.$

(b) $\tan 3x = 0$

$3x = k\pi \Rightarrow x = \dfrac{k\pi}{3}$: $x = 0,\ x = \dfrac{\pi}{3},\ x = -\dfrac{\pi}{3}.$

(c) $\tan 3x = 1$

$3x = \dfrac{\pi}{4} + k\pi \Rightarrow x = \dfrac{\pi + 4k\pi}{12}$: $x = \dfrac{\pi}{12},\ x = \dfrac{5\pi}{12},\ x = -\dfrac{3\pi}{12} = -\dfrac{\pi}{4}$

(d) $\tan 3x = -1$

$3x = -\dfrac{\pi}{4} + k\pi \Rightarrow x = \dfrac{-\pi + 4k\pi}{12}$: $x = -\dfrac{\pi}{12},\ x = \dfrac{3\pi}{12} = \dfrac{\pi}{4},\ x = -\dfrac{5\pi}{12}$

18. $\tan^2 x = 3 \Leftrightarrow \tan x = \pm\sqrt{3}$

For $\tan x = \sqrt{3}$

$x = \frac{\pi}{3} + k\pi = \frac{\pi + 3k\pi}{3}$ $\qquad$ $x = \frac{\pi}{3},\ x = -\frac{2\pi}{3}.$

For $\tan x = -\sqrt{3}$

$x = -\frac{\pi}{3} + k\pi = \frac{-\pi + 3k\pi}{3}$ $\qquad$ $x = -\frac{\pi}{3},\ x = \frac{2\pi}{3}.$

19. $3\sin x = \sqrt{3}\cos x \Leftrightarrow \frac{\sin x}{\cos x} = \frac{\sqrt{3}}{3} \Leftrightarrow \tan x = \frac{\sqrt{3}}{3}$

$x = \frac{\pi}{6} + k\pi = \frac{\pi + 6k\pi}{6}$ $\qquad$ $x = \frac{\pi}{6},\ x = -\frac{5\pi}{3}.$

20. $2\sin^2 x = \sin x \Leftrightarrow 2\sin^2 x - \sin x = 0 \Leftrightarrow \sin x(2\sin x - 1) = 0$

$\sin x = 0$ or $2\sin x - 1 = 0 \Leftrightarrow \sin x = \frac{1}{2}$

(i) in degrees

For $\sin x = 0,\quad x = 0^\circ,\ x = 180^\circ,\ x = 360^\circ$

For $\sin x = \frac{1}{2},\quad x = 30^\circ,\ x = 150^\circ$

(ii) in radians

For $\sin x = 0,\quad x = 0,\ x = \pi,\ x = 2\pi$

For $\sin x = \frac{1}{2},\quad x = \frac{\pi}{6},\ x = \frac{5\pi}{6}$

21. (a) $3\sin^2 x = \cos^2 x \Leftrightarrow 3\sin^2 x = 1 - \sin^2 x \Leftrightarrow 4\sin^2 x = 1 \Leftrightarrow \sin^2 x = \frac{1}{4} \Leftrightarrow \sin x = \pm\frac{1}{2}$

For $\sin x = \frac{1}{2}$ $\quad x = 30°$ or $x = 150°$

For $\sin x = -\frac{1}{2}$ $\quad$ there are no solution in the given interval

(b) $3\sin^2 x = \cos^2 x \Leftrightarrow 3(1 - \cos^2 x) = \cos^2 x \Leftrightarrow 4\cos^2 x = 3 \Leftrightarrow \cos^2 x = \frac{3}{4} \Leftrightarrow \cos x = \pm\frac{\sqrt{3}}{2}$

For $\cos x = \frac{\sqrt{3}}{2}$ $\quad x = 30°$ $\qquad$ For $\cos x = -\frac{\sqrt{3}}{2}$ $\quad x = 150°$

(c) $3\sin^2 x = \cos^2 x \Leftrightarrow \frac{\sin^2 x}{\cos^2 x} = \frac{1}{3} \Leftrightarrow \tan^2 x = \frac{1}{3} \Leftrightarrow \tan x = \pm\frac{1}{\sqrt{3}}\left(= \pm\frac{\sqrt{3}}{3}\right)$

For $\tan x = \frac{1}{\sqrt{3}}$ $\quad x = 30°,$ $\qquad$ For $\tan x = -\frac{1}{\sqrt{3}}$ $\quad x = 150°$

The solutions are $x = 30°,\ x = 150°$

A. Exam style questions (SHORT)

22. $2\cos x = \sin 2x \Leftrightarrow 2\cos x = 2\sin x \cos x \Leftrightarrow 2\cos x\,(1-\sin x) = 0 \Leftrightarrow \cos x = 0$ **OR** $\sin x = 1$

$x = \frac{\pi}{2}, x = \frac{3\pi}{2}, x = \frac{5\pi}{2}$

23. $2\cos^2 x = 2\sin x \cos x \Leftrightarrow 2\cos^2 x - 2\sin x \cos x = 0 \Leftrightarrow 2\cos x(\cos x - \sin x) = 0$

$\cos x = 0, (\cos x - \sin x) = 0$

$\cos x = 0 \Leftrightarrow x = \frac{\pi}{2}$

$\cos x - \sin x = 0 \Leftrightarrow \cos x = \sin x \Leftrightarrow \tan x = 1 \Leftrightarrow x = \frac{\pi}{4}$

24. $2\sin x = \tan x$

$\Rightarrow 2\sin x \cos x - \sin x = 0$

$\Rightarrow \sin x(2\cos x - 1) = 0$

$\Rightarrow \sin x = 0, \cos x = \frac{1}{2}$

$\Rightarrow \sin x = 0, x = \pm\frac{\pi}{3}$

25. $\sqrt{3}\cos x = \sin x \Rightarrow \tan x = \sqrt{3} \Rightarrow x = 60^o$ **or** $x = 240^o$

26. $e^{2x}(\sqrt{3}\sin x + \cos x) = 0$

$e^{2x} = 0$ not possible

$\sqrt{3}\sin x + \cos x = 0 \Leftrightarrow \sqrt{3}\sin x = -\cos x \Leftrightarrow \frac{\sin x}{-\cos x} = \frac{1}{\sqrt{3}} \Leftrightarrow \tan x = -\frac{1}{\sqrt{3}}$

$x = \frac{5\pi}{6}$

27. $\tan^2 2\theta = 1 \Leftrightarrow \tan 2\theta = \pm 1$

For $\tan 2\theta = 1$, $2\theta = \frac{\pi}{4} + k\pi \Rightarrow \theta = \frac{\pi + 4k\pi}{8}$, which gives $\theta = \frac{\pi}{8}$, $\theta = -\frac{3\pi}{8}$

For $\tan 2\theta = -1$, $2\theta = -\frac{\pi}{4} + k\pi \Rightarrow \theta = \frac{-\pi + 4k\pi}{8}$ which gives $\theta = -\frac{\pi}{8}$, $\theta = \frac{3\pi}{8}$

28. $\sin 2x = 2\sqrt{3}\cos^2 x \Leftrightarrow 2\sin x\cos x = 2\sqrt{3}\cos^2 x \Leftrightarrow \sin x\cos x = \sqrt{3}\cos^2 x$

$\Leftrightarrow \cos x(\sin x - \sqrt{3}\cos x) = 0 \Leftrightarrow \cos x = 0$ or $\sin x = \sqrt{3}\cos x$

$\cos x = 0$ gives $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

$\sin x = \sqrt{3}\cos x \Leftrightarrow \tan x = \sqrt{3}$ gives $x = \frac{\pi}{3}$, $x = \frac{4\pi}{3}$

29. (a) $2\cos^2 x + \sin x = 2(1 - \sin^2 x) + \sin x = 2 - 2\sin^2 x + \sin x$

(b) $2\cos^2 x + \sin x = 2 \Rightarrow 2 - 2\sin^2 x + \sin x = 2$

$\sin x - 2\sin^2 x = 0$

$\sin x(1 - 2\sin x) = 0$

$\sin x = 0$ or $\sin x = \frac{1}{2}$

$\sin x = 0 \Rightarrow x = 0$ or π

$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

30. (a) $2\sin^2 x = 1 + \cos x \Leftrightarrow 2(1 - \cos^2 x) = 1 + \cos x \Leftrightarrow 2 - 2\cos^2 x = 1 + \cos x$

$\Rightarrow 2\cos^2 x + \cos x - 1 = 0$

(b) $2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1)$

(c) $\cos x = \frac{1}{2} \Rightarrow x = 60°$, or $x = 300°$

$\cos x = -1 \Rightarrow x = 180°$

Solutions $\Rightarrow x = 60°, x = 180°, x = 300°$

31. $\cos 2x - 3\cos x - 3 - \cos^2 x = \sin^2 x \Leftrightarrow \cos 2x - 3\cos x - 3 = \cos^2 x + \sin^2 x$

$2\cos^2 x - 1 - 3\cos x - 3 = 1 \Leftrightarrow 2\cos^2 x - 3\cos x - 5 = 0$

$\cos x = \frac{5}{2}$ (rejected), $\cos x = -1$

$x = \pi$

32. (a) $4 - \cos 2\theta + 5\sin\theta = 4 - (1 - 2\sin^2\theta) + 5\sin\theta = 2\sin^2\theta + 5\sin\theta + 3$

(b) $2\sin^2\theta + 5\sin\theta + 3 = 0 \Leftrightarrow 2\sin\theta + 3)(\sin\theta + 1) = 0$

$\sin\theta = -1$, $\sin\theta = -\frac{3}{2}$ (rejected)

$\theta = \frac{3\pi}{2}$

33. $3\cos x = 5\sin x \Leftrightarrow \tan x = 3/5$ (=0.6)

$x = 31° + 180°k$

$x = 31°$ or $x = 211°$

34. Either by direct factorisation $(3\sin x - 2)(\sin x - 3)$

OR by setting $y = \sin x$ and solving the quadratic $3y^2 + 4y + 6$

$\sin x = \frac{2}{3}$ or $\sin x = 3$ (rejected)

$x = 42°, x = 138°$

35. (a) $3\sin^2 x + 4\cos x = 3(1 - \cos^2 x) + 4\cos x$

$= 3 - 3\cos^2 + 4\cos x$

(b) $3\sin^2 x + 4\cos x - 4 = 0 \Rightarrow 3 - 3\cos^2 x + 4\cos x - 4 = 0$

$\Rightarrow 3\cos^2 x - 4\cos x + 1 = 0$

By GDC (solving the quadratic $3y^2 - 4y + 1 = 0$)

$\cos x = \frac{1}{3}$ or $\cos x = 1$

(c) By GDC,

EITHER directly by solving the original equation

OR by $\cos x = \frac{1}{3}$, $\cos x = 1$

$x = 70.5°$ or $x = 0°$

36 By using GDC: SolveN

(a) $x = 0$, $x = 1.80$, $x = 2.51$ (3sf) (or exact values $x = 0$, $x = \frac{4\pi}{7}$, $x = \frac{4\pi}{5}$)

(b) $x = 41.8°$, $x = 138°$ (3sf)

B. Exam style questions (LONG)

37. (a) $\sin^3 x = \sin x \Leftrightarrow \sin x(\sin^2 x - 1) = 0 \Leftrightarrow \sin x = 0$ or $\sin^2 x = 1$

$\sin x = 0$ gives $x = 0$, $x = \pi$, $x = 2\pi$

$\sin^2 x = 1 \Leftrightarrow \sin x = \pm 1$ gives $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

(b) $2\sin^3 x = \sin x \Leftrightarrow \sin x(2\sin^2 x - 1) = 0 \Leftrightarrow \sin x = 0$ or $\sin^2 x = \frac{1}{2}$

$\sin x = 0$ gives $x = 0$, $x = \pi$, $x = 2\pi$

$\sin^2 x = \frac{1}{2} \Leftrightarrow \sin x = \pm\frac{1}{\sqrt{2}}$ gives $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{7\pi}{4}$

(c) $4\cos^3 x = \cos x \Leftrightarrow \cos x(4\cos^2 x - 1) = 0 \Leftrightarrow \cos x = 0$ or $4\cos^2 x - 1$

$\cos x = 0$ gives $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$

$4\cos^2 x - 1 \Leftrightarrow \cos^2 x = \frac{1}{4} \Leftrightarrow \cos x = \pm\frac{1}{2}$ gives $x = \frac{\pi}{3}$, $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, $x = \frac{5\pi}{3}$

38. We use double angle formulas to simplify the expressions

(a) $f(x) = 0 \Leftrightarrow \cos 2x + \cos x + 1 = 0$

$\Leftrightarrow 2\cos^2 x - 1 + \cos x + 1 = 0 \Leftrightarrow 2\cos^2 x + \cos x = 0 \Leftrightarrow \cos x(2\cos x + 1) = 0$

$\Leftrightarrow \cos x = 0$ or $\cos x = -\frac{1}{2}$

$\cos x = 0$ gives $x = \frac{\pi}{2}$, $\cos x = -\frac{1}{2}$ gives $x = \frac{2\pi}{3}$

(b) $g(x) = 0 \Leftrightarrow \sin 2x + \sin x = 0$

$\Leftrightarrow 2\sin x\cos x + \sin x = 0 \Leftrightarrow \sin x(2\cos x + 1) = 0$

$\Leftrightarrow \sin x = 0$ or $\cos x = -\frac{1}{2}$

$\sin x = 0$ gives $x = 0,\ x = \pi$ $\cos x = -\frac{1}{2}$ gives $x = \frac{2\pi}{3}$

(c) $f(x) = g(x) \Leftrightarrow \cos 2x + \cos x + 1 = \sin 2x + \sin x$

$2\cos^2 x - 1 + \cos x + 1 = 2\sin x\cos x + \sin x$

$2\cos^2 x - 2\sin x\cos x + \cos x - \sin x = 0$

$2\cos x(\cos x - \sin x) + (\cos x - \sin x) = 0$

$(\cos x - \sin x)(2\cos x + 1) = 0$

$\cos x = -\frac{1}{2}$ or $\cos x - \sin x = 0 \Leftrightarrow \tan x = 1$

$\cos x = -\frac{1}{2}$ gives $x = \frac{2\pi}{3}$ and thus $y = 0$ (since $x = \frac{2\pi}{3}$ is a root of both)

$\tan x = 1$ gives $x = \frac{\pi}{4}$ and thus $y = \sin\frac{\pi}{2} + \sin\frac{\pi}{4} = 1 + \frac{\sqrt{2}}{2}$

Points of intersection $\left(\frac{2\pi}{3}, 0\right)$ and $\left(\frac{\pi}{4}, 1 + \frac{\sqrt{2}}{2}\right)$

39. (a) $3\cos 2x + \sin x = 1 \Leftrightarrow 3(1 - 2\sin^2 x) + \sin x = 1$

$6\sin^2 x - \sin x - 2 = 0$

(b) $(3\sin x - 2)(2\sin x + 1)$

(c) 4 solutions

(d) The last equation gives $\sin x = \frac{2}{3}$ or $\sin x = -\frac{1}{2}$

In the 3rd and 4th quadrants $\sin x$ is negative. Hence $\sin x = -\frac{1}{2}$

$x = -\frac{\pi}{6} + 2k\pi$, so $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

$x = -\frac{5\pi}{6} + 2k\pi$, so $x = 2\pi - \frac{5\pi}{6} = \frac{7\pi}{6}$