ALL LIGHTS AND LIGHTS OUT

An investigation among lights and shadows

SUMA magazine's article by Rafael Losada

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– She has been like this for a week.

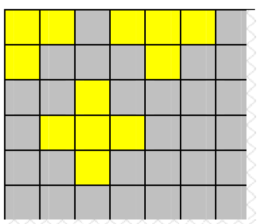
I observed the girl's back bending a little over an enormous flat monitor, where you could see lots of squares forming a tiled board of lights and shadows.

My distressed client, the multimillionaire banker, went on saying: "Cursed be the time when my granddaughter discovered that brainwashing device on the Internet. First, she began with a board divided into 25 squares, and she managed to solve it 5 days ago, although she doesn't really know how she did it. But as soon as she achieved it, the screen was transformed into a huge tiled board with 48 squares in length and width.

– Look! There it is! It's 2,304 altogether. My granddaughter hasn't been able to eat or sleep properly since this monster appeared. You must help me: You'll be highly paid!

– I don't know...! What's the game about?

– It's a devilish invention. *At the beginning all the squares are dark. When you pick up one and press, it changes its colour, that's to say, it lights up if it was dark or it gets dark if it was previously lit up. But the same thing also happens to the squares that are next to the chosen one!* That's why when you manage to light some squares, others that have previously been lit up may go dark.



I was beginning to understand. This case reminded me of the epidemic crazes produced by the Lloyd's 15 or the Rubik's cube: It was about constructing without destroying what you had already achieved.

– And, what's more, according to the puzzle, there's only one way to light all the squares, which is the main objective!

I stretched out my hand to take the piece of paper he was offering to me. It was a € 48,000 cheque to my name. It could be drawn only if I were successful in my mission.

Well! The old man knew how to spend his money! I calculated, with that bitter feeling that invades us when remembering previous hard times, that the dough was equivalent to two years of my salary as a mathematics teacher. Voicing my own thoughts I said to him:

– You seem to consider it a difficult case.

He smiled sadly:

– I have my consultants. They affirm that there are more possible positions in this game than subatomic particles in the whole universe. It can't be easy to find out the only valid one!

THE EASY WAY

I returned to my dingy place with the feeling that at last my life was beginning to improve. In spite of his consultants' opinion, it was evident that the case had been solved beforehand. It would be enough to discover who had put that game in the net. This same person had asserted there was only one solution, so he had to know which one it was!

It didn't seem to be a case where it would be easy to demonstrate the existence of a solution without first constructing a method to obtain it. My intuition for this matter is seldom wrong. It's easy! I'll offer a little bit of dough and I'm sure I'll be given the solution. I can see myself with 48 big ones in my pocket in two days' time.

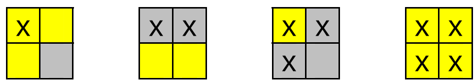
THE DECEPTION

The following day the corpse appeared. He had been cold meat for 48 hours. That bloody number again! All my discoveries led to the same inevitable conclusion: The dead man was the author of the puzzle and the only person in the world who knew the solution. He hadn't left any notes, or records,... nothing. Everything had been in his head... Such a strange fellow! The only thing I could do was to attempt to solve it myself!

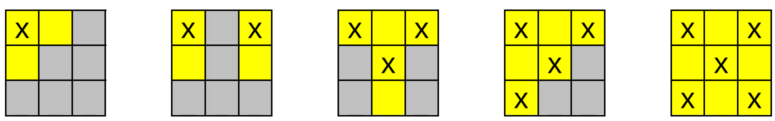
FIRST ILLUMINATION

If I had learnt something in my career as a private investigator and as a mathematician before that, it was that we must begin modestly. So, I drew a tiled board with an only dark square. Good! Now, I select and... there it is... problem solved. It has been lit up. As a kind of joke, I drew a cross in the center of it. "This way I will be able to know which square I have selected", I said to myself smiling.

I tried something "even more difficult". I drew a tiled board with 4 squares. A few attempts were enough to convince me that there was only one solution. At the same time I observed that the order in which I had selected the squares was indifferent.



Until then, everything had gone perfectly well. Next I drew a bigger tiled board. This time I also found the solution very quickly, but it was a bit difficult for me to realize that it was *the only one*, as I had to try all the cases. This took me a long time, because they were 512 altogether! Of course, there were 9 squares now and each one could be chosen or not.



"Things are getting difficult", I thought. Well, at least I have managed to solve it up to now and the solutions found show a strong symmetry. This may help me.

My surprise was great when, little by little, I found one, two, three... sixteen solutions!, in the next square... and not all of them symmetrical.

X		X	
	X		
X		X	X
		X	X

X			X
	X	X	
	X	X	

X	X	X	X
X			X
X	X	X	X

		X	
X			
			X
	X		

By the time I finished, several hours had passed and I was exhausted. I decided to have a rest and it was at that moment when I realized that marking the squares had been a great idea, because this way, the action of "selecting a square" could be properly distinguished from the action of "illuminating a square"!

SECOND ILLUMINATION

I sat up heavily. I had had a nightmare where I could hear furious voices rebuking me endlessly. "You idiot!", they would repeat, "can't you realize that there won't be 16 but 2,304 squares? And each one can or cannot be selected! Have you, by any chance, forgotten the intrinsic difficulty of the exponential problems?"

In my dream I had made a desperate estimation, a hyperastronomical calculation. 2 to the power of 2,304. I applied the decimal logarithm frantically. The logarithm of 2 is slightly greater than 0'3, so the number of possibilities will have... 690 digits more or less! I lost heart. The old man was right.

When I was fully awake I returned wearily to my desk. I could see the tiled board of 16 squares among the scattered sheets of paper –curiously enough, there were as many of them as solutions. But I wasn't even sure that there weren't more solutions. I needed to observe more attentively.

I had been observing unsuccessfully for an hour when I set eyes on a tiled board which had been already solved and that was almost hidden under some papers. It only showed the 1st row, which was lit.

X			

Then, I saw it. The image of the 2nd row appeared clearly in my mind.

		x	x

I picked up the sheet of paper. The mental image and the one on the paper coincided. It was very clear! The 2nd row *IS DEDUCED* from the 1st one, since only by selecting the last two squares of the second row, can all the squares in the first row be lit!

But then, the 3rd row can be deduced from the two previous ones and the same happens to the rest of the rows. *The whole tiled board can be easily deduced just by knowing the position of the crosses in the 1st row!*

I completed the tiled board again, euphorically changing the empty squares and the ones with crosses into zeros and ones.

1	0	0	0
0	0	1	1
0	0	1	1
1	0	0	0

1ª	2ª	3ª	4ª
5ª	6ª	7ª	8ª
9ª	10ª	11ª	12ª
13ª	14ª	15ª	16ª

I hadn't yet finished, when another idea went off in my head: If from the initial position to the final one, *all* the squares have gone through a change of state, it is because each one has been "lit" an *odd* number of times! As "illuminating a square" means "selecting this square or another one next to it", the number of selected squares around a determined one (itself included) must be an odd number. For example, the selected square in 7th is next to two more marked squares, which makes 3 altogether. This number indicates the times that square 7th has gone through a change of state. Therefore, in the end, it will get <u>irremediably</u> lighted.

I was ready to continue! Besides, I knew now that the 16 solutions that I had found were all that existed. It couldn't be otherwise, because the first row, with its four squares, could only have 16 different configurations. And I had discovered out a solution to each of them.

I was rubbing my hands thinking of the money when, paradoxically, the remembrance of my nightmare brought me back to reality. There still existed 2^{48} possible positions for the 1st row of the monstrous board! I grasped my pocket calculator and with trembling fingers I tapped: ... 2 ... x^y ... 48 ... = ... Nearly 300,000,000,000,000 possibilities! Although I would program my computer so as to check one million of states a second in the 1st row –which I found too much for my modest PC– I might have to wait more than eight years before getting to the solution! The old man’s consultants seemed to be right again: the problem hadn’t lost its exponential characteristic. It had only managed to decrease the exponent.

THE GREAT ILLUMINATION

I fell asleep with my head full to the brim with ones and zeros. All night these numbers were dancing wildly in the middle of an empty space only lit by their own brightness. When I was about to wake up, I was in front of a unique gigantic zero that was slowly moving towards my left until it almost disappeared dragging along a trail of light. It looked more like the letter *a* rather than a zero. Then I woke up with a chocked cry: *The algebra*! Instead of experimenting, why not looking for the solution straight away? I had to find an algebraic disposition that showed the problem. I began to work.

I drew again the tiled board with 3 squares in length and width, the one I knew had only one solution, and I allocated one letter for each square in the 1st row.

a	*b*	*c*
x	*y*	
z		

Good! The 2nd and 3rd rows are deduced from the first one, so they must be related to one another. Let’s see: <u>the square taken by the letter *a*</u> (which can be 1 or 0, i.e. can be selected or not) <u>must get illuminated</u>. This means that $a + b + x$ must be an odd number.

It can be expressed like this:

$$a + b + x \equiv 1 \bmod 2$$

(expression read “$a + b + x$ is congruent with 1 modulo 2”,

means that $a + b + x$, as an odd number, leaves a remainder of 1 when divided by 2)

This notation can be simplified if we agree that, from now on, all the mathematical operations will be done with the algebra modulo 2 (where 1+1=0).

This way, we can write $x = a + b + 1$ (verification: $a + b + (a + b + 1) = 1$).

In the same way, to illuminate irremediably the square that is taken by the letter b, the letter y must take such a value that $a + b + c + y$ be "1". From which it can be deduced that the value of y must be precisely $a + b + c + 1$.

In order to illuminate the square taken by the letter x, we will add $a + x + y + 1$, putting the result (which is $a + c + 1$) into the lower square.

Somewhat nervous, I completed the rest of the squares in just a few seconds.

a	b	c
$a + b$ +1	$a + b + c$ + 1	$b + c$ + 1
$a + c$ + 1	0	$a + c$ + 1

This disposition manages to illuminate the first two row of squares regardless of the values of a, b, and c. Oh!, but now *I need the last row to get illuminated, too*! To achieve this I add a fourth "virtual" row whose state, following the same method, will be:

$\boldsymbol{b + c}$ **+ 1**	$\boldsymbol{a + b + c}$	$\boldsymbol{a + b}$ **+ 1**

Good, the 3rd row has already been illuminated, but now there's the fourth row left –the one I had added before–. This row was added to oblige the 3rd one to be lighted but it didn't appear on the original board. That's why none of these "virtual squares" will be able to be selected. All right, they have to eliminated, suppressed, overridden:

$$\begin{cases} b + c + 1 = 0 \\ a + b + c = 0 \\ a + b + 1 = 0 \end{cases}$$

A very simple linear system of three equations! I could smell the money already. I had the dough in my pocket for sure! I began to put the system in order as usual (we mustn't forget we work with modular arithmetic):

$$\begin{cases} \quad\;\; b + c = 1 \\ a + b + c = 0 \\ a + b \qquad = 1 \end{cases}$$

I tried to calm down. First of all, we'd better make sure that it works. From the previous system it can easily be deduced that $a = 1$, $b = 0$, $c = 1$. We substitute these values in the board and... bingo! we've found the only solution to the board of 9 squares!

1	0	1
0	1	0
1	0	1
0	**0**	**0**

ECSTASY AFTER THE GREAT ILLUMINATION

I had found a method for transforming the supposed exponential problem into a simple linear one. The monstrous board wouldn't scare me any more. Finding the solution lay in generating and solving a linear system of 48 equations. With my pen and paper, and thanks to simplicity of the modular arithmetic, it wouldn't take me more than a few hours. I was so sure of being about to solve the problem –and of getting my generous reward– that I decided to spend some time tying up the loose ends before continuing. To begin with, why did the board with 16 squares have more than one solution? I could find it out now with the self-confidence that my method gave me:

a	b	c	d
$a + b + 1$	$a + b + c + 1$	$b + c + d + 1$	$c + d + 1$
$a + c + 1$	d	a	$b + d + 1$
$b + c + d + 1$	$a + b + d + 1$	$a + c + d + 1$	$a + b + c + 1$
0	**0**	**0**	**0**

Here was the reason for so many solutions. The system (?) generated by the added row was indeterminate now and I could see clearly the reason why any disposition of the first row was valid to solve it. Regardless of the values ascribed to the first row, the row added to the last one would always remain dark.

Delighted, I attacked the 25 squares board, the one my client's granddaughter had solved almost by chance.

a	b	c	d	e
$a + b$ $+ 1$	$a + b + c$ $+ 1$	$b + c + d$ $+ 1$	$c + d + e$ $+ 1$	$d + e$ $+ 1$
$a + c$ $+ 1$	d	$a + e$	b	$c + e$ $+ 1$
$b + c + d$ $+ 1$	$a + b + d + e$ $+ 1$	$a + c + e$	$a + b + d + e$ $+ 1$	$b + c + d$ $+ 1$
e	d $+ 1$	c $+ 1$	b $+ 1$	a
$\boldsymbol{b + c + e}$ **+ 1**	$\boldsymbol{a + b + c}$	$\boldsymbol{a + b + d + e}$	$\boldsymbol{c + d + e}$	$\boldsymbol{a + c + d}$ **+ 1**

So the system to be solved is:

$$\begin{cases} b + c + e = 1 \\ a + b + c = 0 \\ a + b + d + e = 0 \\ c + d + e = 0 \\ a + c + d = 1 \end{cases} \Leftrightarrow \begin{cases} a = e + 1 \\ b = d + 1 \\ c = d + e \end{cases}$$

It works! The values of *d* and *e* are free, which means that these unknowns can take any value. Therefore, there must be 4 solutions. Quickly, I represented them (to facilitate the checking and visualization, I decided to cover only the value 1 squares).

1	1			
1	1		1	1
		1	1	1
	1	1	1	
	1	1		1

	1	1		1
	1	1	1	
		1	1	1
1	1		1	1
1	1			

1		1	1	
	1	1	1	
1	1	1		
1	1		1	1
			1	1

			1	1
1	1		1	1
1	1	1		
	1	1	1	
1		1	1	

I wasn't very surprised to find the same basic configuration in all cases, because it was clear that any gyration or symmetry of a solution would produce a new one, as long as it didn't coincide with the original one. I had already observed this fact when finding the 16 solutions in the 16 squares board. I had occurred to me that it wasn't 2^{48} the possibilities of the 1st row of the monstrous board but only 2^{24}, since if there were only one solution it had to be symmetrical! This number of possibilities (with a reversible number 1st row) could be exhaustively analyzed now, testing all the cases with the help of the computer. But it didn't matter any more. The problem had stopped being an exponential one, and what's more, I could even head for the solution straight away.

THE FOURTH ILLUMINATION

Now that I was very close to the solution, I wasn't in a hurry to reach it. Human nature is like this. Besides, there were two reasons for my rejecting the idea of proceeding to draw at once a huge tiled board of 48 squares in length and width and then put the unknowns a_1 ,..., a_{48} in the first row.

The first reason was that I'd rather have the computer doing the mechanical work for me, generating and solving the system, though I didn't know yet how to program it to do such a thing.

The second reason came form my past as a mathematician. Before searching for the solution I'd like to prove that it really exists!

It occurred to me that I might call on one of the methods that announces the *Mathematics Prince*'s name, a well-deserved title, even modest, that goes to the great Gauss (who was then the inventor of the algebra of congruence and creator of the modular notation mentioned above). All in all, I was in the face of a linear equations system. I should be able to analyze it!

Let's see! The system generated by the added row can be expressed like this: A . X = C, where A is a square matrix of the same order as the number of squares in the first row. If all the rows that make up the matrix A are independent of one another, that's a guarantee (because of the existence of the inverse matrix) that the system can be solved, and that there's one solution. This is what happens, for example, when the first row has one, two or three squares.

But what happens when one or more rows are dependent of the rest? Damn it! Then, the system has more than one solution (not infinite solutions as each unknown can only take two values)... or it hasn't any at all! I had to eliminate this last possibility if I wanted to guarantee the existence of a solution in all cases.

Let's go deeper into it! In what cases the incompability, i.e. the non-existence of a solution is produced? It's now when we are in the face of Gauss' method. We must remember that essentially it consists of adding and subtracting the different equations until obtaining a equivalent system where the last equation has only one unknown, the one before the last two, etc.

In this process a row of the matrix A will change into a row of zeros only when there exists a dependency of this row with regard to others. For example, in the case of the system that solves the board of 25 squares, we can observe that the 3rd row is the addition of the 1st one plus the 5th one. Fortunately, this same relation is kept for the corresponding values of the second member. Only when this doesn't happen, i.e. when there isn't the same dependency in the elements of the matrix C, as there is among the rows of the matrix A, will the system lack a solution.

$$\begin{cases} \mathbf{b}+\mathbf{c}+\mathbf{e}=\mathbf{1} \\ a+b+c=0 \\ \mathbf{a}+\mathbf{b}+\mathbf{d}+\mathbf{e}=\mathbf{0} \\ c+d+e=0 \\ \mathbf{a}+\mathbf{c}+\mathbf{d}=\mathbf{1} \end{cases}$$

At last I understand it all! It was like a spark, something physical that made my hairs stand on end. *The interdependence of the elements of the matrix C was* ***exactly*** *the same as the one of the rows of the matrix A.* As it usually happens in these cases, the sudden realization of this reality cannot be properly described or communicated. The only thing I can do is to give the evidence of the veracity of such an affirmation.

Let's see it with an example (which can be generalized in any situation). The following table shows the evolution from the 1st row of the independent terms until we get to the definitive elements of C in the last row. It can be observed that, of course, this table is equivalent to the one already shown, only when giving value "0" to all the unknowns. We can also observe the perfect symmetry in each row.

0	0	0	0	0
1	1	1	1	1
1	0	0	0	1
1	1	0	1	1
0	1	1	1	0
1	**0**	**0**	**0**	**1**

The first "1" of the 2nd row comes from the addition of "1" plus "*a* + *b*" in order to get the square taken by the letter *a* lighted. Whenever we construct a new row, we add "1" in each square, and we go on like this until we get to the row added to the last one, which is the row that generates the system. That's to say, the construction of the matrix A is the result of a series of additions that can be seen as the addition of "1" in matrix C. Or, if we want to be more precise, whenever the coefficient of an unknown is annulated it's because it's been added an even number of times. If all the coefficients are annulated, then the total number of additions has been an even number, which means having added an even number of "1". Consequently, any relation among the squares of the last row *exists both for the coefficients of the unknowns as well as for the corresponding independent terms*.

[It's very important to point out here that the above exposed is the *consequence of adding always a "1" in each square when constructing each row* (trying to light the above square). If any squares had to remain dark (as we'll see later) the compatibility of the system could be jeopardize.]

Conclusion: the board of 48 squares in length and width had a solution and I had to find the procedure to find it. There was only a little problem to solve...

THE FITH ILLUMINATION

Feeling quite pleased with the progress of my investigations, the only thing that remained to be done, was to find a way to get the computer ready to generate the system corresponding to the added row of the board of 48 squares in length and width. In my previous cases I hadn't had the opportunity of facing a similar problem. I knew how to write a program that solved an equations system (for example, by means of the efficient Gauss' method), but how could I design a program that generated *the system itself* before solving it?

I revised my observations on the case repeatedly, till I discovered the solution. The clue to solve it was asking myself a ridiculous question: What's the meaning of the *a* in the first square? Answer: It's just a letter indicating that *IT REFERS TO THE FIRST SQUARE*. That's to say, it shows just a position. But, then, using a positional notation, for example a vector, would be enough to be able to do without letters. The first element of the vector indicates the coefficient of the letter *a*, the second one, the letter *b*, and so on. Incidentally, I add a last element that shows the state of independent term.

(1, 0, 0, 0, 0, **0**)	(0, 1, 0, 0, 0, **0**)	(0, 0, 1, 0, 0, **0**)	(0, 0, 0, 1, 0, **0**)	(0, 0, 0, 0, 1, **0**)
(1, 1, 0, 0, 0, **1**)	(1, 1, 1, 0, 0, **1**)	(0, 1, 1, 1, 0, **1**)	(0, 0, 1, 1, 1, **1**)	(0, 0, 0, 1, 1, **1**)
(1, 0, 1, 0, 0, **1**)	(0, 0, 0, 1, 0, **0**)	(1, 0, 0, 0, 1, **0**)	(0, 1, 0, 0, 0, **0**)	(0, 0, 1, 0, 1, **1**)
(0, 1, 1, 1, 0, **1**)	(1, 1, 0, 1, 1, **1**)	(1, 0, 1, 0, 1, **0**)	(1, 1, 0, 1, 1, **1**)	(0, 1, 1, 1, 0, **0**)
(0, 0, 0, 0, 1, **0**)	(0, 0, 0, 1, 0, **1**)	(0, 0, 1, 0, 0, **1**)	(0, 1, 0, 0, 0, **1**)	(1, 0, 0, 0, 0, **0**)
(0, 1, 1, 0, 1, **1**)	(1, 1, 1, 0, 0, **0**)	(1, 1, 0, 1, 1, **0**)	(0, 0, 1, 1, 1, **0**)	(1, 0, 1, 1, 0, **1**)

[In fact, this table is equivalent to a three-dimensional matrix whose index (i, j, k) express the value of the coefficient of the k-th unknown and the independent term in the square (i, j).]

No more letters! I rush to my computer. I've only got to put in it the 48 vectors of the first row, the method of constructing the other rows starting from the 1st one, and thanks to Gauss it won't take more than one second to give me the long-awaited solution.

LAST SCENE

Happy granddaughter, … satisfied banker…, and as far as I was concerned, as well as getting a large amount of money I had also learnt a lot!

x		x			x	x		x				x		x						x							x						x		x				x		x	x			x		x
	x				x	x	x			x			x			x	x	x				x	x	x	x				x	x	x			x			x			x	x	x				x	
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MORE LUMINOUS GAMES

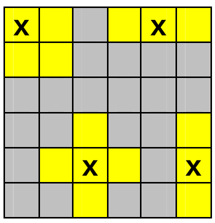
When playing **All Lights**, which means lighting the board as described above, the player selects only the marked squares in the 1st row. The player selects, row by row, all those square *whose square above remains dark* because this is the only way to light them. This procedure involves selecting all the marked squares.

The different "illuminations" of our hero can be applied to many other cases. To begin with, it's obvious that the generalization to any rectangles will not lead to any changes. However, when the widht of the rectangles is greater than their height, the number of operations can be reduced if we rotate the rectangle 90 degrees (this way the number of squares in the first row, and consequently the order of the system, is reduced).

Other aspects of the game can also be changed: the squares altered when one is chosen (Merlin Square, Flip, Rey), the squares that have to be altered (Lights Out and alike), the number of possible states in each square (they can be more than two), and even playing surface (plane, cylinder, Möbius Band, Torus, Klein's Bottle and Cross-Cap). Let's see how these variations affect the solution to the game.

MERLIN SQUARE

In Merlin Square game, the only change we observe is the squares that alter their state when a corner or a square on a side are selected. (Merlin Square will coincide with All Lights when there are no borders, the same will happen when playing on closed surfaces.) To make it clear, the three possible types of alterations are represented in the following board.



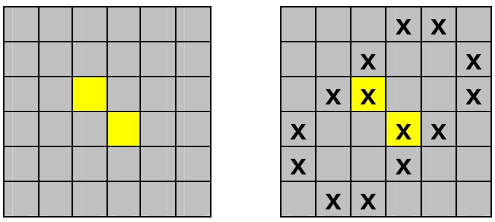
a	*b*	*c*	*d*	*e*	*f*
	s	**s**	**s**	**s**	**s**
					s

It would be enough to make the appropriate modifications by constructing each row, starting from the 1st one, to solve the question. However, bearing in mind the special behaviour of the squares on the corners and the sides, this time the system is not generated by adding a new virtual row, but by the conditions imposed on the squares marked with “**s**”. A new case comes up when the number of columns is 2. In this case, the state of the square corresponding to the bottom right corner of the rectangle is independent of the states of the squares in the 1st row, and therefore a 3rd unknown placed on that corner must be added.

Lights Out in Internet

LIGHTS OUT

Of the many variations of this type of game, it is perhaps Lights Out the most surprising for the player. There are even fan clubs for it, the first of which was created in 1996 and can still be found at www.mit.edu/~kbarr/lo/. The way the state of the squares is altered is identical to All Lights. However, as its name implies, the lights must be put out. It’s easy to understand that if the whole board was lit, there wouldn’t be any essential differences with All Lights. But in the initial position of Lights Out, *only some squares of the board appear lit*, and these are the lights that must be put out. Therefore, we can consider All Lights as a particular case of Lights Out. As an example, the following chart presents a proposed problem and its only solution.



It has already been said that All Lights always has a solution since that *all* squares must be altered. Besides, when the generated system is undetermined, the Gauss’ method will generate one or more rows of zeros in the coefficient matrix and the corresponding zeros in the matrix of independent terms. There will be two solutions for each of these rows. The following chart shows the number of solutions (the gyrations and symmetries are considered different) depending on the dimensions of the rectangular board, as far as the order 20.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1	2	1	1	2
2	2	1	4	1	2	1	4	1	2	1	4	1	2	1	4	1	2	1	4	1
3	1	4	1	1	8	1	1	4	1	1	8	1	1	4	1	1	8	1	1	4
4	1	1	1	16	1	1	1	1	16	1	1	1	1	16	1	1	1	1	16	1
5	2	2	8	1	4	1	16	2	2	1	16	1	2	2	16	1	4	1	8	2
6	1	1	1	1	1	1	1	64	1	1	1	1	1	1	1	1	64	1	1	1
7	1	4	1	1	16	1	1	4	1	1	128	1	1	4	1	1	16	1	1	4
8	2	1	4	1	2	64	4	1	2	1	4	1	128	1	4	1	2	1	4	64
9	1	2	1	16	2	1	1	2	256	1	2	1	1	32	1	1	2	1	256	2
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	2	4	8	1	16	1	128	4	2	1	64	1	2	4	256	1	16	1	8	4
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
13	1	2	1	1	2	1	1	128	1	1	2	1	1	2	1	1	8192	1	1	2
14	2	1	4	16	2	1	4	1	32	1	4	1	2	16	4	256	2	1	64	1
15	1	4	1	1	16	1	1	4	1	1	256	1	1	4	1	1	16	1	1	4
16	1	1	1	1	1	1	1	1	1	1	1	1	1	256	1	256	1	1	1	1
17	2	2	8	1	4	64	16	2	2	1	16	1	8192	2	16	1	4	1	8	128
18	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
19	1	4	1	16	8	1	1	4	256	1	8	1	1	64	1	1	8	1	65536	4
20	2	1	4	1	2	1	4	64	2	1	4	1	2	1	4	1	128	1	4	1

Nevertheless, the situation is complicated further when we play Lights Out. If the solution is unique for All Lights, it’s also unique for any initial states of Lights Out because in both cases the coefficient matrix will be nonsingular, regardless of what may happen to the matrix of independent terms. Oh! So what happens in those rectangles that admit several solutions in All Lights? In Lights Out, not all squares will alter their state, but only those that have been lit in the initial situation. Therefore the regularity will be broken in such a way that when the rows of zeros appear in the coefficient matrix, *the corresponding independent term may or may not also be zero*. Consequently, for every row of zeros in coefficient matrix, there’s a one in two possibility (and probability) that the corresponding equation may be an identity.

Conclusion: **the previous chart not only shows the number of solutions of All Lights (and of Lights Out if there were any) but also the inverse ones of the probability that there may exist a solution in Lights Out.**

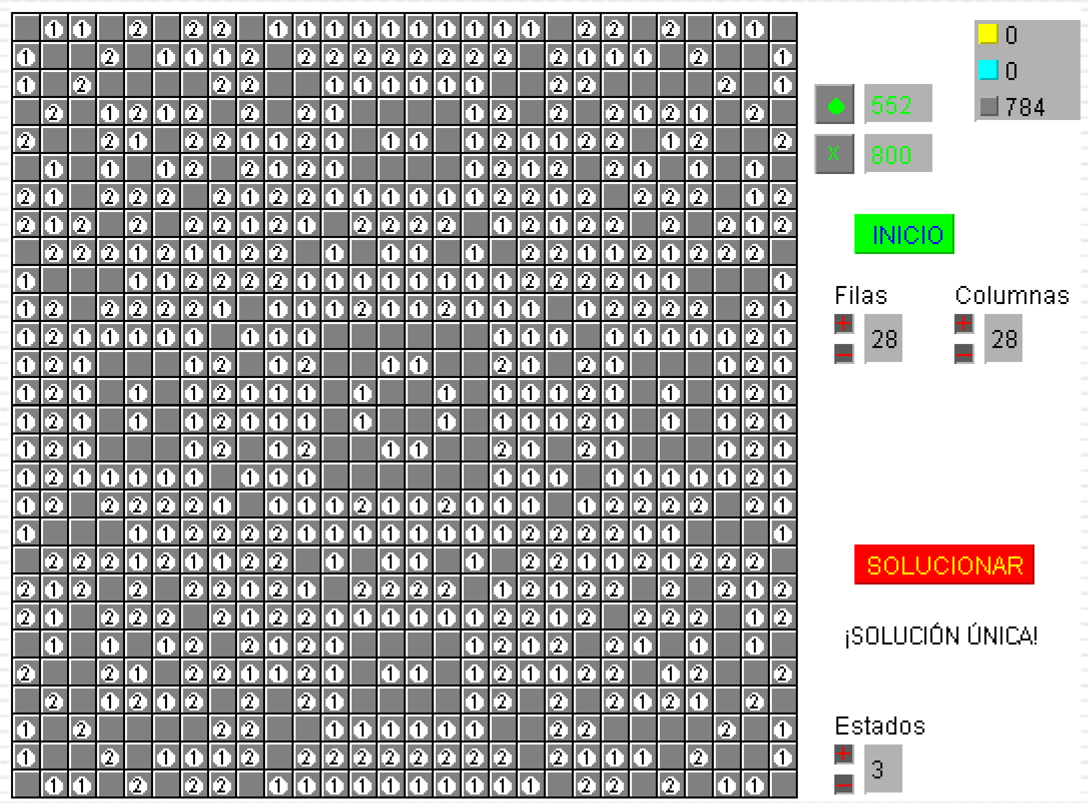
To set an example: If we light at random the squares of a board of 5 squares each side, the probability that they may be put out is ¼. If the board has 19 each side, this probability is reduced to one in 65536!

ENLARGING THE RANGE OF COLOURS

This type of games can also be generalized so that each square admits **more that two states** (i.e. there is an intermediate stage between fully lit and fully dark).

Those rectangles in which there are a greater number of solutions are the more interesting ones to play due to the added challenge of finding the most economical, i.e. the one that selects the fewest squares to solve the problem. Increasing the arithmetic modulo also means increasing the power base that generates the number of solutions, and thus there are soon have millions of solutions. *Before finding the solutions*, it is extremely difficult, if not impossible, to decide which one will be the most economical.

In order to generate the system corresponding to several states, taking the number of possible states as modulo is enough.



The only solution to All Lights with 3 states in a board of order 28

Nevertheless, we must be careful with the modular arithmetic. Our hero observed that the system generated in modulo 2 by All Lights is always compatible *because if two rows of the* *coefficient matrix are dependent that means they are the same, and this equality is kept by the independent terms*. However, this is not the same in other modulo. The simplest example can be found in the order 2 board of All Lights modulo 3. There is no solution now! Let's see the reason. The system generated is:

$$\begin{cases} a + 2b = 2 \\ 2a + b = 2 \end{cases}$$

(Now, it's a good time to stop, have a cup of tea and check if this is really the system.) The determinant of the coefficient matrix is –3, which is equivalent to 0 in modulo 3! It follows that the apparently determinate system happens to be incompatible. It must be observed that the 2nd row of the coefficient matrix is twice the 1st one, and this relation is not kept by the independent terms now.

And what's more, everything gets rather complicated when, in addition to this, the modulo used is a compound number. In these cases, *there are proper divisors* of 0. For example, using modulo 6, equations as 3a = 0, have 3 solutions: 0, 2, and 4. This characteristic invalidates the Gauss' method, as we no longer have the guarantee that the solutions found using his method are really solutions of the system.

We mustn't forget this method often needs to multiply the two members of an equation by one number. But if this number *is divisor of the modulo* we risk eliminating or reducing the relation that exists between the unknowns. In these cases it would seem logical to resort to less drastic processes rather than the elimination of coefficients. A good way could be using the substitution method. It's a pity this method is far less elegant and also rather bothersome. We can solve the problem by factorizing the modulo in prime modulos, as long as, in addition to this, they are relative primes (i.e. different primes). Thus, in modulo 6, the system has solution S_6 on condition that it has solution S_2 in modulo 2 and solution S_3 in modulo 3. The following identity allow us to relate the solutions between these modulos:

$$S_6 = (3\,S_2 + 4\,S_3) \bmod 6$$

Let's see one example to clarify things. We want to solve the following problem in Lights Out modulo 6, where the numbers represent the initial illumination, i.e. the number of alterations each square must experience to be put out.

1	1
0	4

The corresponding system is generated:

a	*b*
5a + 5b + 1	*5a + 5b* + 1
***a + 2b* + 4**	***2a + b* + 2**

Then we have that the module 6 system is:

$$\begin{cases} a + 2b = 2 \\ 2a + b = 4 \end{cases}$$

The determinant of coefficient matrix is –3, which is not equivalent to 0 in modulo 6, any more! Thus, we can presume the system has one solution. Only… it hasn't! We cannot jump to this conclusion because when –3 is proper divisor of 0 the inverse matrix doesn't exist (the inverse of 3 doesn't exist).

What we must do is solve the previous system using modulo 2 (one solution), using modulo 3 (three solutions), and obtaining with given identity the three modulo 6 solutions of the system.

$$S_6(1) = 3\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$S_6(2) = 3\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$S_6(3) = 3\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

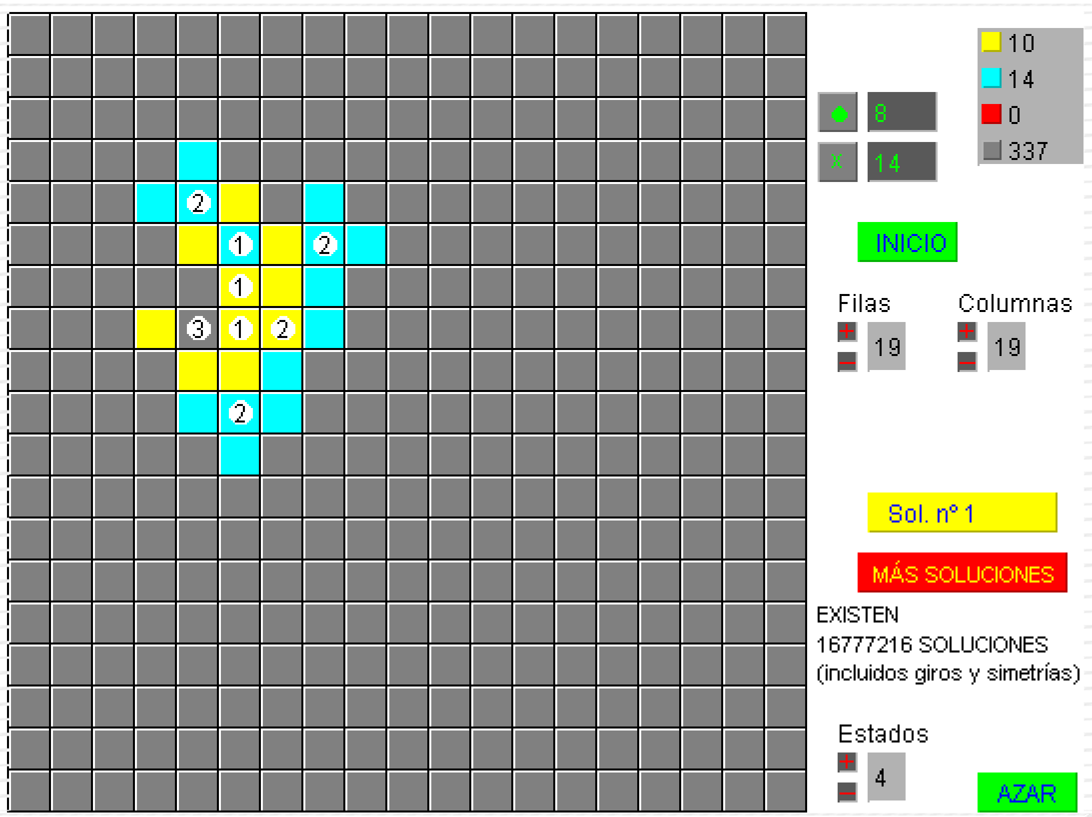
Therefore, the three solutions to the problem are (the numbers represent now the number of times each square must be selected in order to put out the whole board):

2	0
5	5

0	4
3	3

4	2
1	1

And what happens if the modulo is 4, for example, which is the product of equal primes? This is the most complicated case. Our only option is to change the way in which the coefficients are eliminated when applying the Gauss' method, since if we multiply a row by 2 we risk eliminating information and causing the appearance of false "solutions". As a possible way out, we could look for the odd coefficients in the matrix (or submatrix which is being reduced) and force the process to choose only these ones as pivots swapping columns if necessary. If all the submatrix coefficients were even numbers, we then proceed to divide all the coefficients of the unknowns in each row by two (once we have found a solution, the process is inverted so as not to loose solutions).



A Lights Out problem with 4 states, and millions of solutions

MESSAGE IN A BOTTLE FOUND IN INTERNET

Subject: **Light's Out** Date: Sun, **2 Jan 2000** 15:03:30

From: **Gary Watson** To: kbarr

Hi,

I wrote a program similar to Tiger's "Light's Out" back in 1985 or so, in GWBasic for IBM PC's. It was called "Flip". I deliberately used the block characters so that people without a graphics card could still play it. I'm not sure, but it's possible I invented the game as I had never heard of it before hand. My version was different from the hand held Tiger game (which is pretty cool by the way) in that the object was to get all the squares lit up, always five squares were lit up at random to start the game, and you were not allowed to press an illuminated square. I'm not sure why I made that restriction, but it made it hard to solve. The starting position was generated randomly, and a mathematician friend of mine speculated that some starting positions were insolvable, but he couldn't prove it (I suspect it was sour grapes because he couldn't ever beat it). Tiger has a patent on the Light's Out game, and it would be a hoot if it turned out that the invention really belonged to the public domain! I uploaded Flip to about a hundred BBS's from 1985 to about 1988 or thereabouts. Any idea how long Tiger has been making their game?

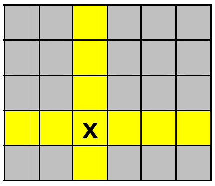
Gary Watson. Technical Director. Nexsan Technologies, Ltd.

We had already seen that, in fact, only one of each 4 dispositions at random in a 5x5 board has a solution in Lights Out. Restricting the cases and taking into account only those in which 20 squares have to put out (to illuminate, in Watson's case) alters slightly this probability to 13,326 cases out of 53,130.

Pocketsize Lights Out game patented by Tiger

ANOTHER FLIP

After reading the above message, it’s curious to observe that the name Flip can currently be found on the Internet in relation to a puzzle that is different from Lights Out and Watson’s game, though with obvious similarities –the squares that alter their state are those that are **in the same row or column** as the selected square.



This change radically alters the method of resolution because each row isn’t constructed based on the previous one. We must approach this question now, taking complete rows and columns instead of isolated squares.

Solving Flip in its version All, i.e. altering *all* the squares in the rectangle, is very easy. The only thing we have to do is to select all the squares in any row or column, as long as they are an odd number. If the rectangle has even dimensions, the only solution will be selecting all the squares.

The problem gets complicated in the version *Out*, i.e. when we must put out *only some previously illuminated squares*. As I have mentioned before, the strategy must be based on complete rows and columns. And, of course, we mustn’t forget that we are still working with easy modulo 2.

I will use the following notes:

Data	
n	Number of rows in the rectangle.
m	Number of columns in the rectangle.
B	Matrix whose elements come to 1 in the squares that must be put out and 0 in the others.
$r_B(i)$	Addition (mod 2) of the elements of the row i of the matrix B.
$c_B(j)$	Addition (mod 2) of the elements of the column j of the matrix B.
t_B	Addition (mod 2) of the elements of the matrix B.
Unknown	
A	Wanted matrix, whose elements come to 1 in the squares that must be chosen and 0 in the others.
Parameters	
$r(i)$	Addition (mod 2) of the elements of the row i of the matrix A.
$c(j)$	Addition (mod 2) of the elements of the column j of the matrix A.
t	Addition (mod 2) of the elements of the matrix A.

In the first place, we have to main equation that relates the matrix we start from to the wanted matrix A:

$$\boldsymbol{A(i,j) = r(i) + c(j) - B(i,j)} \qquad \forall i, j \qquad \text{(mod 2)}$$

To understand this equation, we'd better set an example. Let's suppose we have to put out square (1,1), i.e. B(1,1) = 1. Every time the 1st row or the 1st column changes their state, i.e., every time r(1) and c(1) alter their value, the square (1, 1) will change its colour. If the 1st row and the 1st column change an odd number of times [r(1) + c(1) = 1], the square (1,1) shouldn't be selected [A(1, 1) = 1-1 = 0]. Analogically, this happens in the rest of the cases.

The following three equations (mod 2, although I'm keeping the negative signs in order to facilitate their understanding) can be deduced from the main equation by adding the elements in each row and column. These three equations will lead us to the solution to the problem in no time:

$$\begin{cases} r(i) = \sum_{j=1}^{m} A(i,j) = m \cdot r(i) + t - r_B(i) & \forall i \\ c(j) = \sum_{i=1}^{n} A(i,j) = n \cdot c(j) + t - c_B(j) & \forall j \\ t = \sum_{i=1}^{n} r(i) = m \cdot t + n \cdot t - t_B \end{cases}$$

Depending on the parity of *m* and *n*, four cases arise with the corresponding solution:

i) $m=0$ mod 2, $n=0$ mod 2. The only solution is (mod 2):

$$\boldsymbol{A(i,j) = -r_B(i) - c_B(j) - B(i,j)} \quad \forall i, j$$

ii) $m=0$ mod 2, $n=1$ mod 2. This implies $t = c_B(j) \quad \forall j$. Thus, there will be a solution only when all the columns have the same parity in the number of the initially lit squares. Whenever this requirement of consistency is accomplished, which will happen once in every 2^{m-1}, there will be 2^{m-1} solutions:

$$\boldsymbol{A(i,j) = -r_B(i) + c_B(j) - B(i,j) + c(j)} \quad \forall i, j$$

with $c(m) = c_B(m) - \sum_{j=1}^{m-1} c(j)$ where $c(j)$ can take any value.

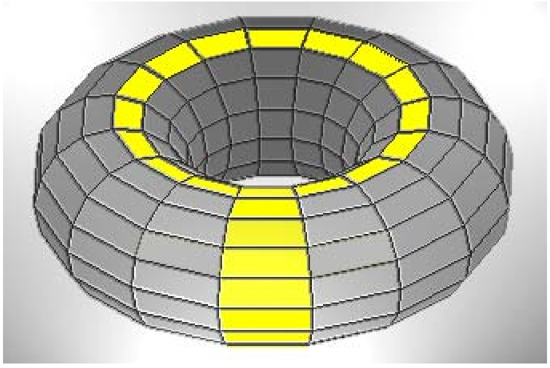
iii) $m=1$ mod 2, $n=0$ mod 2. This case is the same as the previous one, except for the fact that the rectangle is gyrated 90 degrees.

iv) $m=1$ mod 2, $n=1$ mod 2. This implies $r_B(i) = c_B(j) = t \quad \forall i, j$. Thus, there will be a solution only when all the rows and columns have the same parity in the number of the initially lit squares. Whenever this requirement of consistency is accomplished, which will happen once in every 2^{m+n-2}, there will be 2^{m+n-2} solutions:

$$\boldsymbol{A(i,j) = r(i) + c(j) - B(i,j)} \ \forall i, j$$

with $r(n) = r_B(n) - \sum_{i=1}^{n-1} r(i)$; $c(m) = c_B(m) - \sum_{j=1}^{m-1} c(j)$ where $r(i)$, $c(j)$ can take any value.

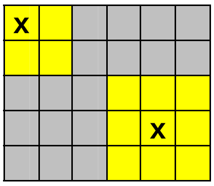
Anyway, the matrix A has been found at the same time as the method of discussion about the existence of a solution has.



Playing Flip on a Torus

REY (KING)

I have named a variant of Lights Out “Rey”. Curiously enough, I haven’t seen it circulating yet. It may be the first time it appears here… and yet it’s such an obvious possibility! Now, the squares that change their state, in addition to the chosen one, are those that surround it and the same happens with de squares threatened by the king of chess (which explains the name of the game).



The same as with Flip, it’s very easy to solve King in its version *All*, but things get complicated in the version *Out* (though it’s still easier to solve than Lights Out).

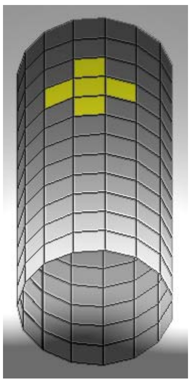
The analysis of King leas us to a system with as many equations and unknowns as squares are found in the 1st row and (if more than one column exists) the rest of the 1st column. The reason is that the state of the squares marked with the first nine letters in the following board determines the state of the rest of the squares. Consequently, these ones are the unknowns now.

a	*b*	*c*	*d*	*e*	
f					**s**
g					**s**
h					**s**
i					**s**
	s	**s**	**s**	**s**	**s**

Analogically to what our hero did, adding the “virtual” squares marked with “**s**” is enough to settle the corresponding equations that allow the darkened squares to get illuminated. On the other hand, when increasing the number of unknowns, the rectangles to which there are lots of solutions are plenty. For example, we achieve more than a thousand million of solutions in an order 8 rectangle (like a chessboard) with 4 possible states in each square.

OTHER SURFACES

Up to now, all the games were developed on the plane. But, this is not the only surface that we can use as a game board. For example, we may imagine we design the board on a cylindrical surface, in such a way that there is no first or last one any more. It won't be necessary to complicate the flat representation of the board. The only thing we mustn't forget is that the "lost column" represented is beside (and preceding) the "first column". It isn't difficult to change from the flat model to the real spatial one, since we have just to imagine that the left and the right sides or borders of the rectangle have been "stuck". Of course, if we leave these borders alone and stick the top border to the bottom one, we'll obtain a cylindrical surface again.



Sticking two opposite borders of the board.

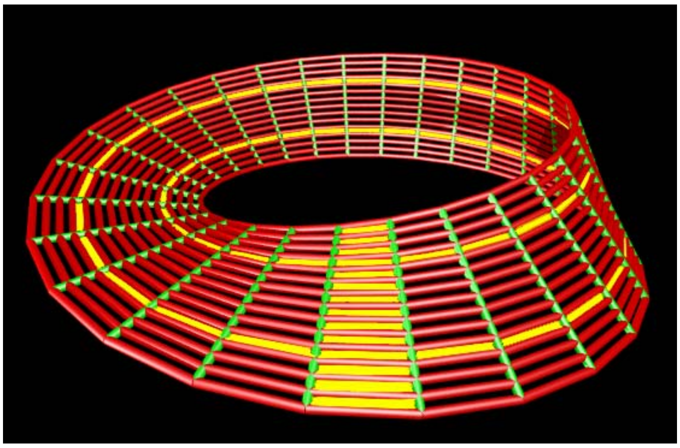
When sticking borders, there are 5 different ways of doing it:

[Corners disappear:]

1. You stick two opposite borders: ***Cylinder***. If the columns are still straight lines, the rows are transformed into rings. There are two borders left.
2. You stick two opposite borders after turning over (gyrating) one of them: ***Möbius' Band***. If the columns are still straight lines, the rows are changed into Möbius' bands. There's one border left (!).

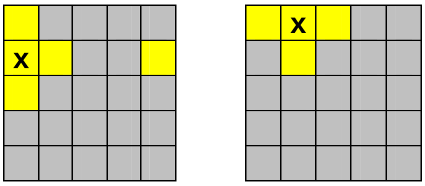
[The borders disappear as well:]

3. You stick the opposite borders two by two: ***Torus***. Both the rows and the columns are rings.
4. You stick two opposite borders as in the Cylinder and the other two as in the Möbius' Band: ***Klein's Bottle*** (it can only be constructed in a more than 3 dimensions space). If the rows are transformed into rings, the columns become Möbius' bands.
5. You stick the opposite borders two by two as in Möbius' Band: ***Cross-Cap*** (it can only be constructed in a more than 3 dimensions space). Both the rows and the columns are Möbius' bands.

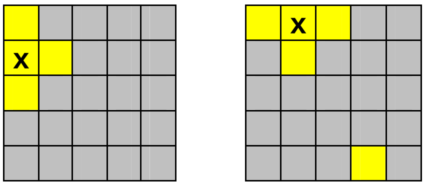


The selection of a square produces this effect on Flip over Möbius' Band

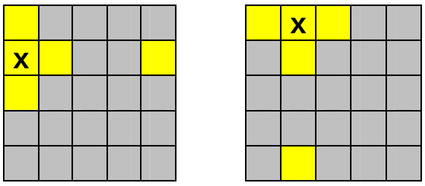
To make myself clear, in the following boards we can see the effects these surfaces have on the illumination of some squares of the rectangle in the game of All Lights (or in Lights Out).



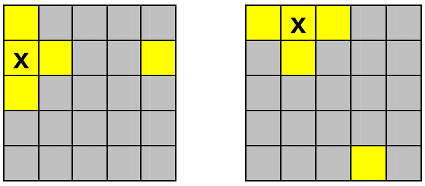
CILINDRO



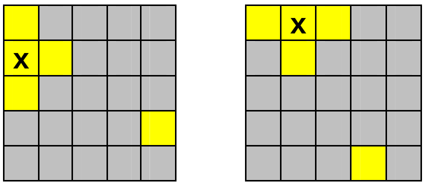
BANDA DE MÖBIUS



TORO



BOTELLA DE KLEIN



GORRO CRUZADO

PLAYING ON A CYLINDER

The system is generated essentially in the same way as on the plane, only taking into account the vicinity of the first and last columns.

PLAYING ON A MÖBIUS' BAND, ON A TORUS, OR ON A KLEIN'S BOTTLE

The system generated duplicates –in general[1]– its order, as the last row cannot be reduced from the previous ones now. Therefore, the squares of this last row become new unknowns. The conditions imposed to the last two rows generate the system.

a	*b*	*c*	*d*	*e*
f	*g*	*h*	*i*	*j*

s	**s**	**s**	**s**	**s**
s	**s**	**s**	**s**	**s**

Although we cannot build it, we can play on a Klein's bottle

In Flip's case, it's obvious that the game doesn't undergo any modifications when choosing the torus (or the cylinder) as a surface. It we play on a Möbius' band, or a Klein's bottle, the number of solutions is spectacularly increased due to the coincidence of each column with its opposite. This allows any values given to the squares of half the columns to be valid, when there's a solution.

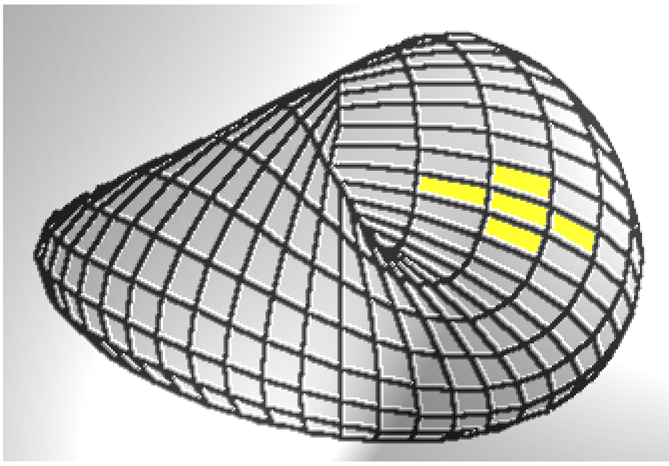
[1] For rectangles of less than 3 rows or columns the order of the system can be occasionally reduced.

PLAYING ON A CROSS-CAP

The system generated is –in general– of order 2(R+C–3), where R and C indicate the number of rows and columns of the rectangle. This is due to the fact that now all the border squares in the rectangle, except for two corners that can be deduced, are unknowns. The system is generated by the conditions imposed on the last two rows and the first and last columns, except for the two corners mentioned above.

a	*b*	*c*	*d*	*e*
f				*g*
h				*i*
j				*k*
	l	*n*	*p*	

s				**s**
s				**s**
s				**s**
s	**s**	**s**	**s**	**s**
	s	**s**	**s**	



The Cross-Cap can also be used as a game board.

In the case of Flip, the number of solutions is increased even more than when a Möbius' Band was chosen. Now also the opposite rows coincide. Consequently, the squares in these rows can also take any value.

YOU DON'T BELIEVE A SINGLE WORD UNLESS YOU CHECK IT (and that's the right thing to do)

In the Web page www.anarkasis.com/rafa/ you can find the Java applet that I have designed so that we can propose, solve and check any problems (between 2 and 7 states) of All Lights, Merlin Square, Lights Out, etc., up to order 50. This limitation is only due to the restrictions imposed by the size of the screen. When large boards are chosen, it's spectacular to find that the solution appears at once[2]. If there is more than one solution *all of them* can be seen. (In fact, I have decided on a maximum of a thousand million solutions: if somebody is interested in seeing also the following ones, please, keep in touch with me *when you have seen all the solutions the program shows*.)

The core of the program corresponding to Lights Out (with **w** possible states in each square) is based on the following loops, made on a **M** column rectangle where matrix **b** take values between 1 and w**-**1 in the squares that must be put out and 0 en the others. Its expression in Java is like this[3]:

// Calculation **coefficients** of the unknowns `a(k)` *first and last* of the row `R`

```
int R1=R-1; int R2=R-2; int N=M+1;
for (int k=1; k<N; k++) {
       a[R][1][k]=(-a[R2][1][k]-a[R1][1][k]-a[R1][2][k])%w;
       a[R][M][k]=(-a[R2][M][k]-a[R1][M][k]-a[R1][M-1][k])%w;
}
```

// Calculation **independent terms** `a(M+1)` *first and last* of the row `R`

```
a[R][1][N]=(-a[R2][1][N]-a[R1][1][N]-a[R1][2][N]+b[R1][1])%w;
a[R][M][N]=(-a[R2][M][N]-a[R1][M][N]-a[R1][M-1][N]+b[R1][M])%w;
```

// Calculation all the others coefficients and terms of the row R

```
for (int C=2; C<M; C++) {
       for (int k=2; k<N; k++) {
              a[R][C][k]=(-a[R2][C][k]-a[R1][C-1][k]-a[R1][C][k]-a[R1][C+1][k])%w;
       }
       a[R][C][N]=(-a[R2][C][N]-a[R1][C-1][N]-a[R1][C][N]-a[R1][C+1][N]+b[R1][C])%w;
}
```

THE END

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[2] The button Random included in the program that solves Lights Out (and alike) has been conceived to settle at random only initial solvable positions. On the contrary, if we decide to settle a specific initial position, there may not be a solution (the program will let us know).

[3] The symbol % in Java indicates the modulo.