PROJECTILE MOTION *Air resistance*

The force due to air resistance can be modeled as either first- or second order in the velocity. For relatively small objects moving in air at modest velocities, we can use first order, and that will be the approach here. However, just to show what happens, consider the second-order differential equations. We have a force which opposes the motion along the current direction of the velocity vector (which of course is continuously changing). So, in components, we have from Newton's second law (the mass has been divided out and incorporated into the constant k)

$$\frac{d^2x}{dt^2} = -k\, v(t)^2 \cos(\phi(t)) \qquad \frac{d^2y}{dt^2} = -g - k\, v(t)^2 \sin(\phi(t))$$

where both the magnitude v and direction ϕ of the velocity are functions of time. Since v depends on both the x- and y-component of velocity, these differential equations are coupled. So we have a pair of second-order, nonlinear, coupled differential equations. We can re-write these as

$$\frac{d^2x}{dt^2} = -k\, v(t)\, \frac{dx}{dt} = -k\, v(t)\, v_x \qquad \frac{d^2y}{dt^2} = -g - k\, v(t)\, \frac{dy}{dt} = -g - k\, v(t)\, v_y$$

since

$$\frac{dx}{dt} = v(t)\, \cos(\phi(t)) = v_x \qquad \frac{dy}{dt} = v(t)\, \sin(\phi(t)) = v_y$$

To show the coupling more explicitly, we can also write

$$\frac{dv_x}{dt} = -\left(k\sqrt{v_x^2 + v_y^2}\right) v_x \qquad \frac{dv_y}{dt} = -g - \left(k\sqrt{v_x^2 + v_y^2}\right) v_y$$

These ODE's are now first order (in the velocities), but are still coupled, and nonlinear. So we aren't going to make much progress with this, analytically. (Of course these are readily solved numerically.)

If we use a first-order air resistance, proportional to the velocity, then the ODE's become much simpler:

$$\frac{d^2x}{dt^2} = -k\, v(t)\, \cos(\phi(t)) = \frac{dv_x}{dt} = -k\, v_x \qquad \frac{d^2y}{dt^2} = -g - k\, v(t)\, \sin(\phi(t)) = \frac{dv_y}{dt} = -g - k\, v_y$$

These are readily solved, using a convolution integral, or Laplace transforms, integrating factors, etc. The velocity results are

$$v_x(t) = v_x(0)\, \exp(-k\, t) = v_0\, \cos(\theta)\, \exp(-k\, t)$$

$$v_y(t) = \frac{-g}{k}\,(1 - \exp(-k\, t)) + v_y(0)\, \exp(-k\, t) = \frac{-g}{k} + \left(\frac{g}{k} + v_0\, \sin(\theta)\right) \exp(-k\, t) \qquad (1)$$

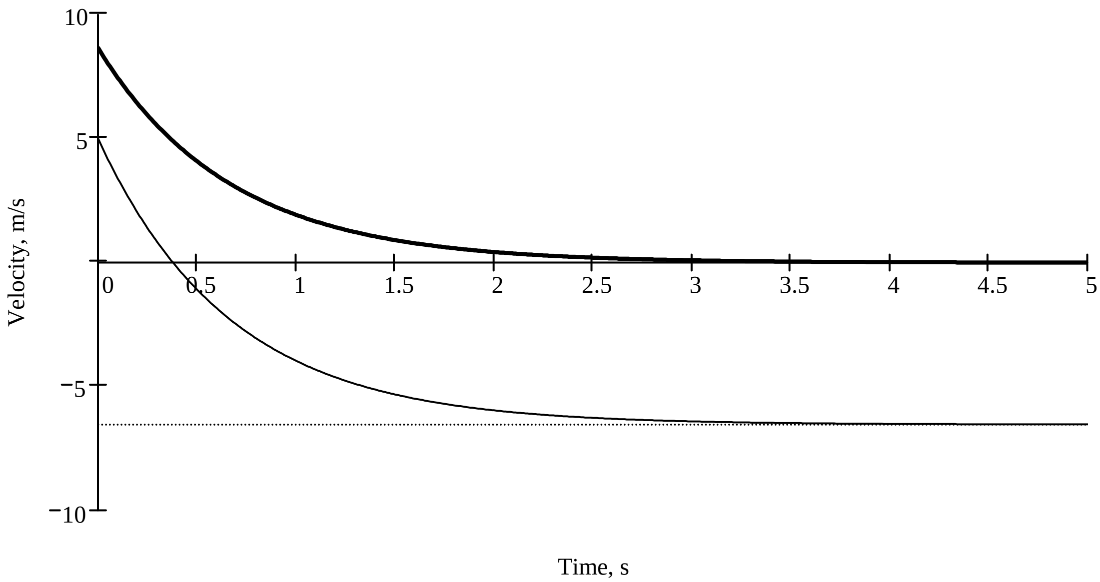
Note that the constant k depends on both the drag coefficient and the mass. The larger the mass, the *smaller* is k, and the larger the drag coefficient, the larger is k. So, for a heavy object, the constant k is small, and the effect of drag is minimal.

Next we explore the behavior of these velocity components as the constant k becomes small.

$$\lim_{k \to 0} \left(v_0 \cos(\theta) \exp(-k\,t)\right) = v_0 \cos(\theta)$$ x velocity, no drag

$$\lim_{k \to 0} \left[\frac{-g}{k} + \left(\frac{g}{k} + v_0 \sin(\theta)\right) \exp(-k\,t)\right] = v_0 \sin(\theta) - g\,t$$ y velocity, no drag

so we have the expected results from the no-drag situation. Note that k cannot be exactly zero. Next is a plot of these velocity components, for k = 1.5. The x velocity (bold line) approaches zero, and the y velocity approaches a constant, negative, nonzero value, as we can see by inspection of Eq(1). Thus we will have a terminal velocity of -g / k. This can of course be seen immediately from the ODE for the y-velocity (when the derivative is zero, we have the "terminal" condition).



Next we attempt to find the time of maximum y, by setting the y velocity to zero; this gives:

$$Tmax(k) := \frac{-1}{k} \ln\left(\frac{1}{1 + \frac{v_0\,k}{g} \sin(\theta)}\right) \tag{2}$$

Checking the limit again, we recover the no-drag result for the time of maximum y (the vertex):

$$\lim_{k \to 0} \left(\frac{-1}{k} \ln\left(\frac{1}{1 + \frac{v_0\,k}{g} \sin(\theta)}\right)\right) = \frac{v_0}{g} \sin(\theta)$$

To find the x and y positions as a function of time, we integrate the velocities; the initial position in x is taken to be zero:

$$x(t,k) := \frac{v_0 \cos(\theta)}{k} (1 - \exp(-k\,t)) \qquad y(t,k) := \left[\frac{1}{k}\left(\frac{g}{k} + v_0 \sin(\theta)\right)(1 - \exp(-k\,t)) - \frac{g\,t}{k} + y_0\right] \tag{3}$$

Again we take limits to see if the no-drag results are recovered:

$$\lim_{k \to 0} \left[\frac{v_0 \cos(\theta)}{k} (1 - \exp(-k\,t)) \right] \qquad v_0 \cos(\theta)\,t$$

$$\lim_{k \to 0} \left[\frac{1}{k} \left(\frac{g}{k} + v_0 \sin(\theta) \right) (1 - \exp(-k\,t)) - \frac{g\,t}{k} + y_0 \right] \qquad \frac{-1}{2} g\,t^2 + v_0 \sin(\theta)\,t + y_0$$

and all is well. Now we can find the position of the maximum height, by using the time from Eq(2) in the position equations x(t) and y(t). This gives:

$$\mathrm{xmax}(k) := \frac{v_0^{\,2} \sin(\theta) \cos(\theta)}{g + v_0 \sin(\theta)\,k} \qquad \mathrm{ymax}(k) := y_0 + \frac{v_0 \sin(\theta)}{k} + \frac{g}{k^2} \ln\left(\frac{1}{1 + \frac{v_0\,k}{g} \sin(\theta)} \right) \tag{4}$$

Check limits:

$$\lim_{k \to 0} \frac{v_0^{\,2} \sin(\theta) \cos(\theta)}{g + v_0 \sin(\theta)\,k} \qquad \frac{v_0^{\,2}}{g} \sin(\theta) \cos(\theta)$$

$$\lim_{k \to 0^+} \left(y_0 + \frac{v_0 \sin(\theta)}{k} + \frac{g}{k^2} \ln\left(\frac{1}{1 + \frac{v_0\,k}{g} \sin(\theta)} \right) \right) \qquad y_0 + \frac{v_0^{\,2}}{2\,g} \sin(\theta)^2$$

and these are the results for the no-drag case.

Finally, a parameter of interest is the time of flight. We get this by setting y(T) = 0 and solve for T. In this situation, however, we have a transcendental equation, which cannot be explicitly solved for T.

$$\frac{1}{k} \left(\frac{g}{k} + v_0 \sin(\theta) \right) (1 - \exp(-k\,T)) - \frac{g\,T}{k} + y_0 = 0$$

We can try to expand the exponential in a series of only a few terms, which will be powers of T:

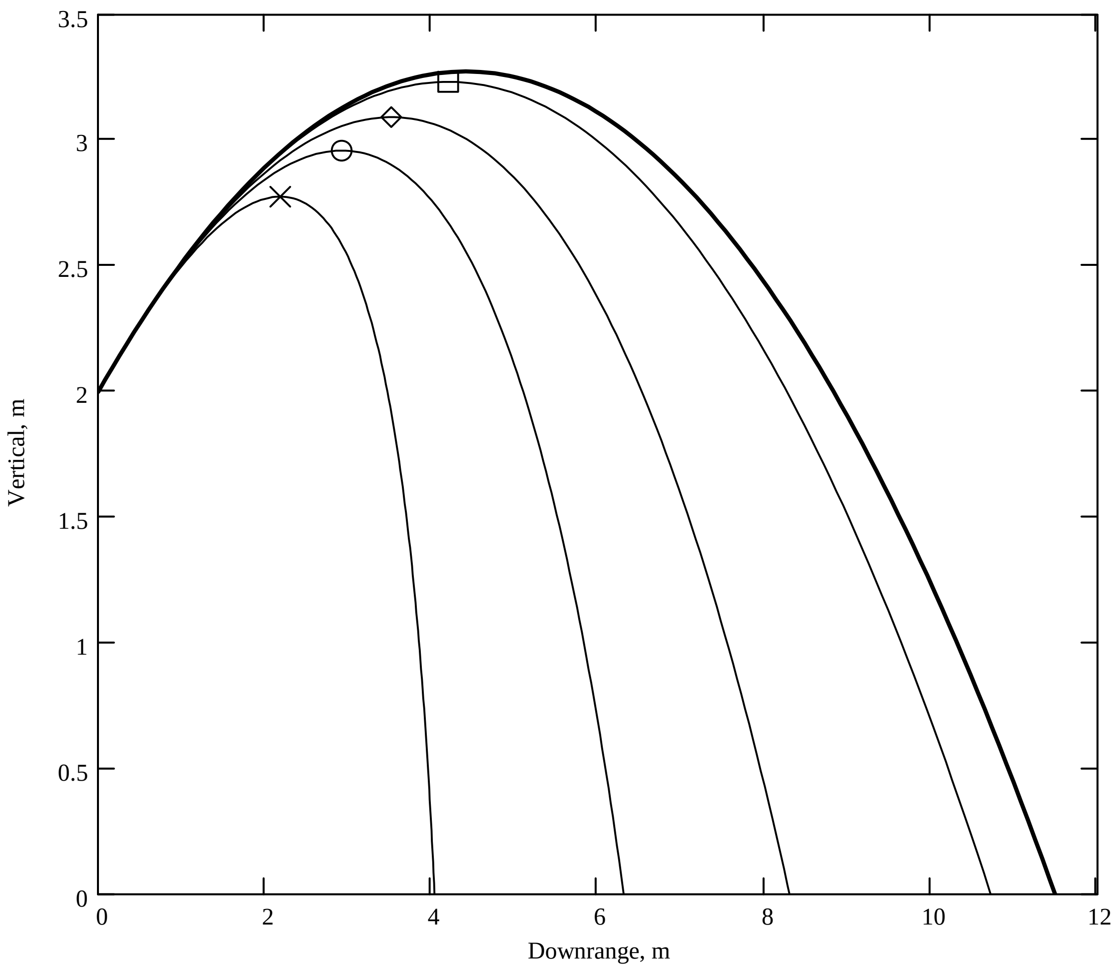
$$y_0 + v_0 \sin(\theta)\,T + \frac{-1}{2} k \left(\frac{g}{k} + v_0 \sin(\theta) \right) T^2 = 0$$

This quadratic can be solved for T,

$$T(k) := \frac{1}{g + k\,v_0 \sin(\theta)} \left[v_0 \sin(\theta) + \sqrt{v_0 \sin(\theta) \left(v_0 \sin(\theta) + 2\,k\,y_0 \right) + 2\,g\,y_0} \right]$$

which, remarkably, can be seen to be the same as the non-drag result for TOF as k becomes small. This is because the approximation becomes better as k is smaller. For larger k this TOF will not be especially accurate.

Next we plot the trajectory for a few example cases, indicating the maxima, and checking the final x position (i.e., the range) calculated using x(T), compared to the observed value. For this value of T the y position should be close to zero.



In the plot, the initial velocity is 10 m / s, at an initial angle of 30 degrees. The k values are 0.001 (thick line, essentially no drag), 0.1, 0.5, 1, and 2. Here are the calculated ranges and y positions:

$$d := 0.1 \qquad x(T(d), d) = 10.37 \qquad y(T(d), d) = 0.345$$

$$d := 0.5 \qquad x(T(d), d) = 7.361 \qquad y(T(d), d) = 1.217$$

$$d := 1 \qquad x(T(d), d) = 5.337 \qquad y(T(d), d) = 1.734$$

$$d := 2 \qquad x(T(d), d) = 3.398 \qquad y(T(d), d) = 2.121$$

We see that the approximate TOF is not very good unless the drag coefficient k is quite small. Since the TOF is not very useful, there's little point in using it in the x(t) to find a range.