NOTES ON CIRCULAR MOTION

What is the difference between angular and linear velocity?

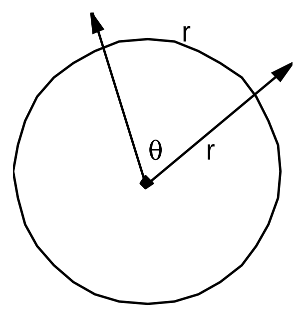
Given a circle, we can pick any point along the circumference as a starting point. Then, as an object moves along the circle we can measure its position as the angle (positive counterclockwise, usually) between its current position and that starting point. The rate at which this angle changes is the angular speed; if we consider that the rotation could be in either the positive or negative direction, then we can use the term angular velocity. Note that *all points along a radial from the center to the circumference will have the same angular velocity*.

The linear velocity for circular motion is a bit more complicated. Clearly the object is not moving in a straight line, so how can there be a linear velocity? For this we need to consider the idea of "arc length" which for a circle is just the portion of the circle's circumference that is traversed by the object over some time interval. Then, using calculus, we can prove that the magnitude of the instantaneous linear velocity v at any time is

$$v = r\,\omega \qquad (1)$$

where ω is the angular velocity and r is the radius of the circle. We see from Eq(1) that the linear velocity increases with the radial distance r from the center of the circle, and we should remember that the direction of this vector quantity is tangent to the circle, in the same direction as the rotation. The key here is that the position angle, which we usually call θ, and its rate of change ω, must be measured in radians and radians per second.

So, what is a radian?



Measuring angles in degrees is arbitrary, from a mathematical point of view. We have 360 degrees, why not 100 or 1000 degrees in a circle? To eliminate this ambiguity a more robust measure of angle was developed. *A radian is that angular measure such that the arc length of a circle subtended (cut off) by the angle is equal to the radius of the circle.* This turns out to be about 57 degrees, and it can be proved that there are 2π radians in 360 degrees. Thus one complete rotation, or to be more fussy, "revolution" of the object around its circular path is 2π radians.

What is uniform circular motion?

When an object moves with a constant speed in a circle, this is called uniform circular motion. However, even if the linear *speed* is constant, the linear *velocity* of the object is continually changing. That velocity at any instant is tangent to the circle, so that the direction of this "tangential" velocity changes as the object moves around the circular path. Since velocity is a vector, this change in direction represents an acceleration, and we know that an acceleration must be caused by a force.

What is "centripetal force"?

If an object is moving on a path that is not a straight line, then there must be a component of acceleration that is perpendicular to the path, and thus there must be a force acting on the object that has a component perpendicular to the path. We call this force the "centripetal" force; the word means "to seek the center." Since the tendency of a moving object is to move in a straight line (inertia), a force must continually be applied that turns the object away from this straight line. A centripetal force must be caused by an identifiable agent, and this force causes the acceleration (change in direction) of the moving object.

Is "centripetal force" a new kind of force?

No, this is just the name we use to describe whatever force is causing an object to move in a path around some central point, usually the center of a circle. For example, if we whirl an object on a string in a circle overhead, parallel to the ground, the string tension provides the centripetal force. A car going around a curve is deflected from its straight line path by the friction force of the tires on the road. Planets and satellites move in curved orbits due to the centripetal force supplied by gravity.

Why do I feel a pull on the string when whirling an object overhead?

The centripetal force is directed inward, toward the center of the circle, which in this case is your hand, holding the string. If the force is directed toward your hand, why does it feel like the string is trying to pull away? Doesn't this mean that there is another force, a "centrifugal" (center-fleeing; think "fugitive") force that is pulling the object away?

The pull that you feel is due to Newton's Third Law; action/reaction. You are applying a centripetal force that acts on the string, which in turn acts on the object, but there is a reaction force that the object applies to the string. This is the pull that you feel. But this is not a separate force, it is a **reaction** force.

To see this, consider what would happen if there was a separate centrifugal force acting on the object. Wouldn't it just cancel out the centripetal force, so that there would be no net force, and the object would just move in a straight line? Or think of it this way-- if you suddenly let go of the string while whirling the object, which way does it go? If you turn off the centripetal force, wouldn't the centrifugal force take over and accelerate the object off in a direction along the radial where it was when you released the string, since that's presumably the direction of action of the centrifugal force?

But this doesn't happen-- the object will fly off in a direction tangent to the circle at the moment the string was released. The reaction force is turned off when the action (centripetal) force is turned off. Now no force acts on the object, so it moves in a straight line, according to Newton's First Law.

Ok, but why do I feel like a force is pushing me into the car door when going around a curve?

Again, it's tempting to say "centrifugal force" pushes you into the door. But what is really happening is that your inertia wants you to keep going in a straight line, *at every instant* as you go around the curve. The car says, no, we want to go to the left, so it applies a centripetal force that pulls you away from the straight line you want to travel. This "battle" between your inertia and the car feels to you like a force pushing you to the right, into the door, but actually the door is pushing you to the left around the curve. This is an example of a "fictitious" force. It is an illusion caused by the fact that your coordinate system (the car) is accelerating. This is called a non-inertial reference frame.

Can we calculate the centripetal force on an object in uniform circular motion?

If we consider the instantaneous (tangent) velocity vectors at two closely-spaced times, and do a bit of geometry, and then a bit of calculus, we can prove that the centripetal acceleration is

$$a_{radial} = \frac{v^2}{r} \tag{2}$$

We use the subscript "radial" to indicate that this acceleration is directed along a radial line connecting the object at its current position to the center of the circle. *For uniform circular motion this acceleration is constant.* Be aware that, contrary to intuition, there does not need to be motion in the direction of an acceleration. To get the force we just use Newton's Second Law, and multiply by the mass of the object

$$F_{centripetal} = \frac{mv^2}{r} \tag{3}$$

The acceleration and force are of course vectors; in these equations we have used their magnitudes. The acceleration and centripetal force both are directed inward, toward the axis of rotation.

Does the centripetal force do any work?

Since the centripetal force is a radial force, it is by definition at right angles to the motion, at all times. Thus there is no component of this force applied along the direction of motion, so no work is done. If there was any tangential component of this force, it would do work, since it would cause an acceleration and change the speed and kinetic energy of the object. This would no longer be uniform (i.e., constant-velocity) circular motion.

What if I don't know the linear velocity?

If we know the time T it takes for a complete revolution (or rotation), then we can use the simple relation

$$v = \frac{2\pi r}{T} \quad \Rightarrow \quad a_{radial} = \frac{4\pi^2 r}{T^2} \tag{4}$$

What is the "round room" and how is it related to centripetal force?

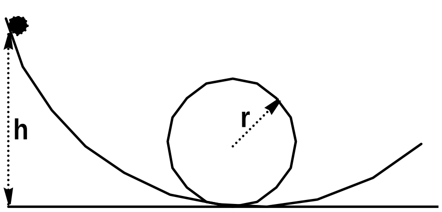
Mr. Evans used to enjoy a ride at Six Flags in Atlanta; the ride was also at Ocean City, long ago. This is a cylindrical "room" that spins around. You enter and stand along the wall, the room starts to spin, and gets going pretty fast. At Ocean City this was on a pier, and after a while of spinning, the floor drops out and you see the water down below. Cool. How did this work-- why don't the people fall into the water?

It's tempting to say, as most people do, that "centrifugal force" keeps us from falling. But we know better, since there is no such thing as centrifugal force. What happens is the centripetal force creates a normal force from the wall on the backs of the people standing against the wall, so that

$$F_{wall} = \mu F_{Normal} = \mu \frac{m v^2}{r} = m g \quad \Rightarrow \quad v = \sqrt{\frac{g r}{\mu}}$$

When the speed of the room is at least this value, no one will fall. Note that the mass (weight) of the people doesn't matter, but their coefficient of friction does. The friction force is directed upward, and this (we hope) cancels the weight force downward.

How about the "ball-around-the-loop" demo we did in class?



For this we would like to calculate the initial height *h* needed to get the ball to go around the circular loop, of radius *r*. This will not be uniform circular motion, because the ball will decelerate as it goes up the loop, and accelerate as it comes back down. So the best way to approach this is to use conservation of energy.

We know that the ball will have potential energy at the initial height, and since we release it from rest, this is all the available energy. As it comes down the ramp it has increasing kinetic energy, and then it enters the loop, with some velocity. The metal loop provides the centripetal force that causes the ball to move in a circular path, while gravity acts to affect the (instantaneous) velocity-- so, the ball slows down as it rises.

The key is to see that at the top of the loop we must have a certain velocity to keep the ball on the track, and this critical velocity v_c is found by equating the centripetal force to the gravitational force:

$$\frac{m v_c^2}{r} = m g \qquad \Rightarrow \quad v_c = \sqrt{g\, r}$$

If the ball has at least this velocity at the top of the loop, it should go around. Now we need to find the initial height needed to attain this critical velocity; we use conservation of energy:

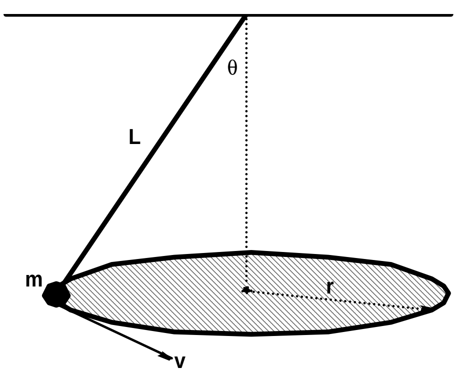
$$m g h \; + \; 0 \; = \; m g \, y(t) \; + \; \frac{1}{2} m v(t)^2$$

Here we have shown the vertical position *y* and the velocity both as functions of time, since they will vary as the ball moves. The critical position is at $y = 2r$, which is the top of the loop. Then

$$m g h \; = \; m g (2r) \; + \; \frac{1}{2} m v_c^2 \quad \Rightarrow \quad h = 2.5 r$$

This says that the ball should be one-half the radius of the loop above the loop. Actually, this analysis ignores the rotational kinetic energy of the ball (it spins, rather than sliding), so the height turns out to be 2.7*r*. Even this isn't complete, since there are other losses that haven't been considered. So if we place the ball at a height of about 3*r*, this should get the ball around the loop.

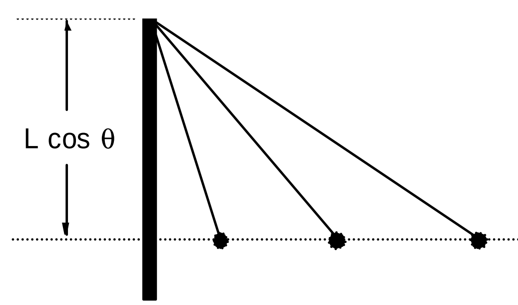
What is a conical pendulum?



If we start a pendulum moving in a circle, as indicated in this sketch, then we have a so-called conical pendulum (since the line L traces out a cone as it moves around). The forces acting on this pendulum are just gravity and the tension of the string. But, the tension in the string has a component directed toward the center of the circle ($T \sin\theta$), in addition to the component that balances the weight mg ($T\cos\theta$). The central component provides the centripetal acceleration needed to keep the pendulum moving in a circle. It can be shown that the period of oscillation of the conical pendulum is

$$\tau \; = \; 2\pi \sqrt{\frac{L \cos\theta}{g}}$$

Interestingly, all the circular pendula in the sketch to the right have the same period, even though their lengths are different! The view is from the side; the lower dotted line is the plane in which the circular motion occurs.



What about orbital motion: planets and satellites?

That is a logical next step- to combine centripetal force with gravitational force. Explaining the motions of the planets was a primary motivator for Newton and his predecessors. We must mention the name "Kepler" since his amazing work was a strong foundation for Newton, in this area. The subject of orbital mechanics is complex, and requires some advanced mathematics. We won't have time this semester to pursue this interesting material.