PROJECTILE MOTION *Targeting*

We seek to find a launch angle θ that will hit a target located at a given position (X,Y), for a given initial velocity. Begin with the Cartesian version of the parabolic trajectory (developed elsewhere):

$$y(x) = \frac{-g}{2\left(v_0 \cos(\theta)\right)^2} x^2 + x \tan(\theta) + y_0 \qquad (1)$$

which can be written as

$$Y = \frac{-g}{2 v_0{}^2}\left(1 + \tan(\theta)^2\right) X^2 + X \tan(\theta) + y_0$$

using the identity

$$\frac{1}{\cos(\theta)^2} = 1 + \tan(\theta)^2$$

Since the unknown is the angle, define

$$\psi = \tan(\theta) \qquad \text{and} \qquad \alpha := \frac{-g}{2 v_0{}^2}$$

then we have a quadratic in (a function of) the angle:

$$Y = \alpha\left(1 + \psi^2\right) X^2 + X \psi + y_0 \qquad \left(\alpha X^2\right)\psi^2 + (X)\psi + \left(y_0 - Y + \alpha X^2\right) = 0$$

$$a := \alpha X^2 \qquad b := X \qquad c := y_0 - Y + \alpha X^2$$

and the solution(s) will be

$$\psi_1 := \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \psi_2 := \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

depending on the discriminant D. We have three cases.

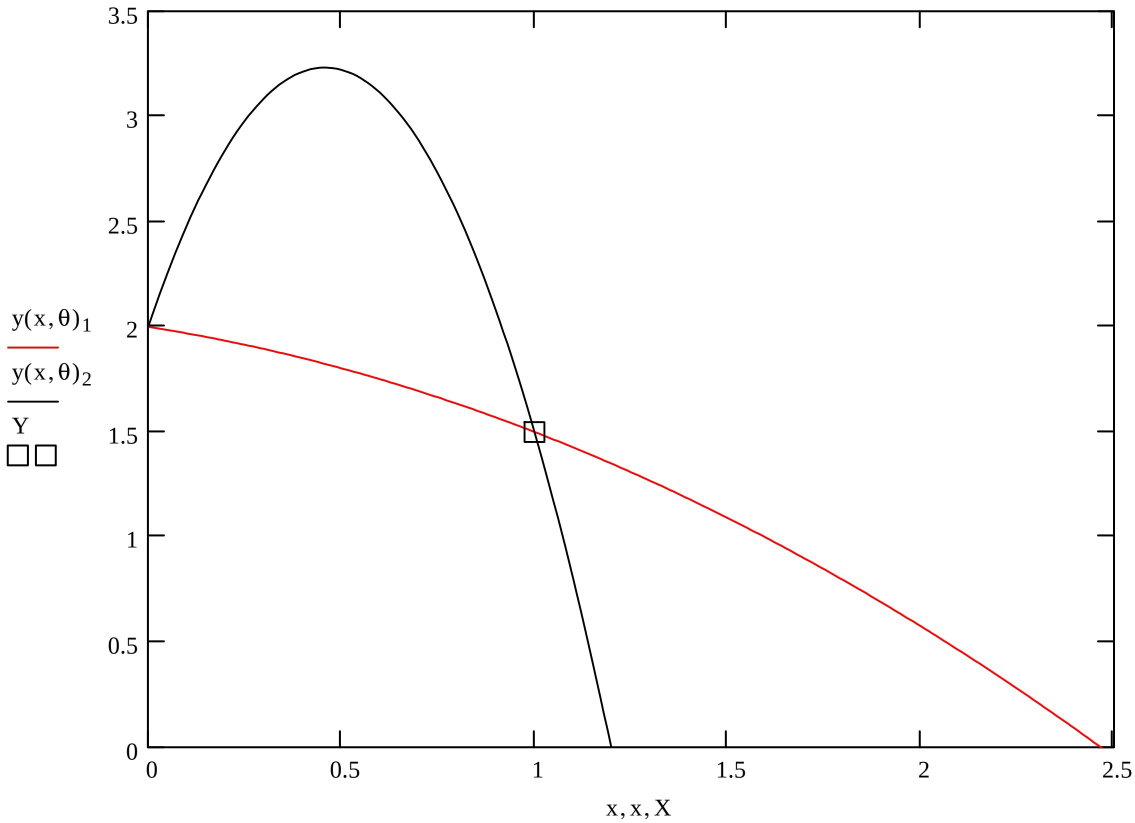
	$D > 0$	target can be hit, two angles
$D = b^2 - 4ac$	$D = 0$	target can just be hit, one angle
	$D < 0$	target cannot be hit

When there is a solution we use the "aim angle" θ in Eq(1): $\theta := \operatorname{atan}(\psi)$

Next we have a plot of an example of this solution, for a general case, with a nonzero initial height. Note that there are two solutions, one downward (negative angle) and one upward. Clearly these will have different TOF's and ranges. The values for this example are shown below, using expressions derived elsewhere.

$$\theta\,\frac{180}{\pi} = \begin{pmatrix} -16.054 \\ 79.489 \end{pmatrix} \qquad T(\theta) = \begin{pmatrix} 0.513 \\ 1.314 \end{pmatrix} \qquad R(\theta) = \begin{pmatrix} 2.466 \\ 1.198 \end{pmatrix}$$

The parameters are: initial velocity 5 m/s; initial height, 2 m. The solution also works for zero initial height.



Of interest is the so-called "**parabola of safety**" which is an envelope of the points that can just be hit by a single angle. If a point is outside this envelope, it cannot be hit (and so is "safe"). Consider the discriminant when it is zero (single solution, i.e., a double root). Then we can write

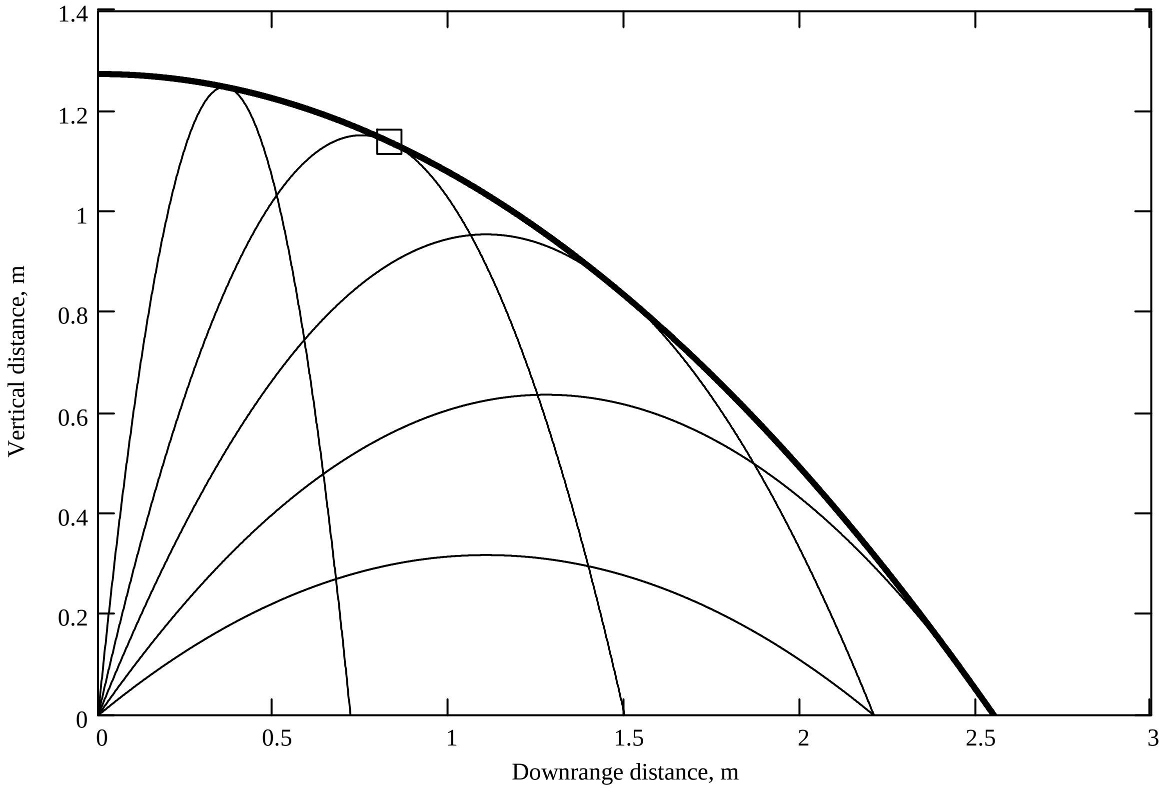
$$b^2 = 4\,a\,c \qquad \text{so that we have} \qquad X^2 = 4\left(\alpha\,X^2\right)\left(y_0 - Y + \alpha\,X^2\right)$$

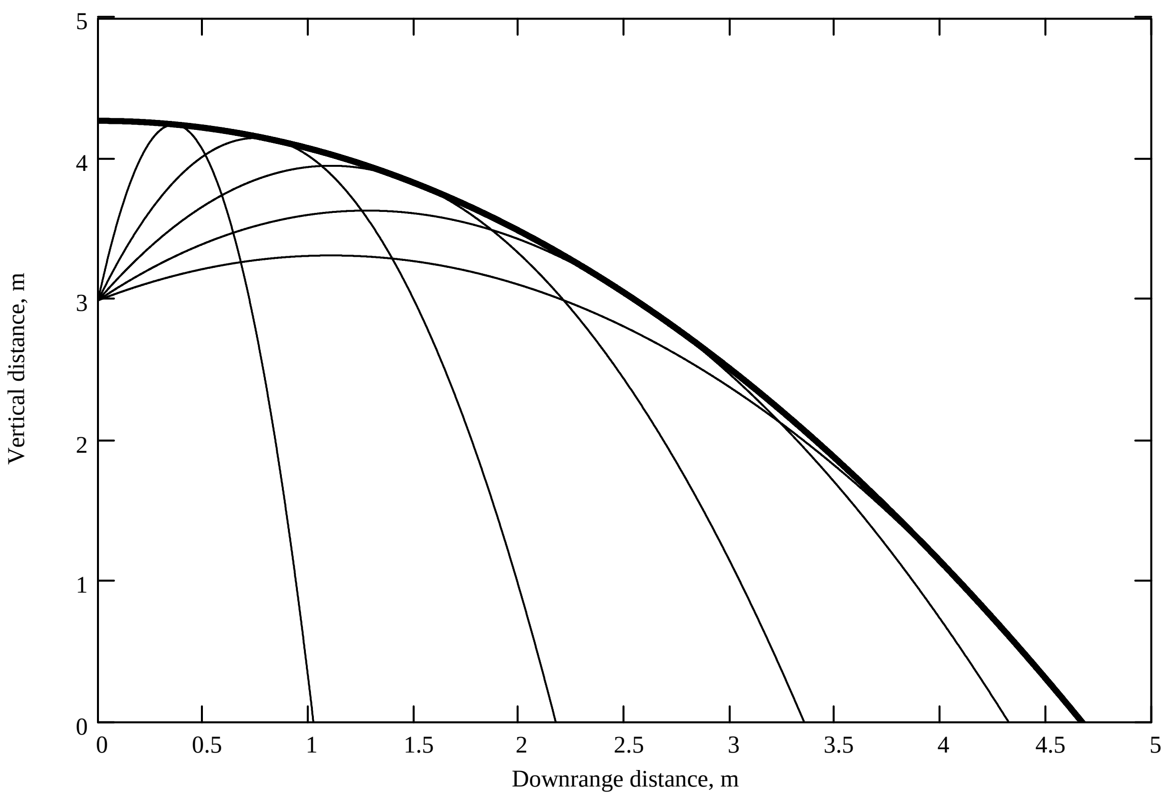
which we want to solve for Y in terms of X, to see what form this envelope has. This is

$$Y = \alpha\,X^2 + y_0 - \frac{1}{4\,\alpha} \qquad\qquad Y = \frac{-g}{2\,v_0^{\,2}}\,X^2 + y_0 + \frac{v_0^{\,2}}{2\,g} \tag{2}$$

Next are plots of this parabola (thick lines) with some example trajectories shown. Note that several are just tangent to the envelope, while others are entirely inside. The idea is that a point located outside this parabola cannot be hit by a projectile launched with the given initial velocity, *no matter at what angle*. This would be the case with, say, an antiaircraft gun (an artillery piece is correctly called a "gun") with a fixed muzzle velocity. Any target outside this parabola cannot be engaged.

Note that the envelope also works for a trajectory that starts at a nonzero initial height.


1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
Vertical distance, m
Downrange distance, m


5
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Vertical distance, m
Downrange distance, m

An interesting fact is that the velocity vector for a trajectory at the tangent point with the envelope is perpendicular to the initial velocity vector. We can show this by taking the derivative of the envelope, Eq(2):

$$\frac{dY}{dX} = \frac{-g}{v_0^{\,2}}\,X$$

and we need to evaluate this at the point of tangency with the trajectory. At this x-point the tangents (derivatives w.r.t. x) are equal. Writing the trajectory Eq(1) as a quadratic in the usual manner (re-using a,b,c) we have

$$y(x) = a\,x^2 + b\,x + c \qquad \text{so that} \qquad \frac{dy}{dx} = 2a\,x + b$$

and the latter is equal to the derivative of the envelope, so

$$2\,a\,X_{tangent} + b = \frac{-g}{v_0^{\,2}}\,X_{tangent} \qquad \text{from which} \qquad X_{tangent} = \frac{-b}{2\,a + \frac{g}{v_0^{\,2}}}$$

Now with $a = \dfrac{-g}{2\left(v_0\,\cos(\theta)\right)^2}$ $\quad b = \tan(\theta)$ $\quad$ then we have $\boxed{X_{tangent} = \dfrac{v_0^{\,2}}{g\,\tan(\theta)}}$

Next we evaluate the derivative of the envelope at this point, so

$$\frac{dY}{dX}\left(X = X_{tangent}\right) = \frac{-g}{v_0^{\,2}}\,\frac{v_0^{\,2}}{g\,\tan(\theta)} = \frac{-1}{\tan(\theta)}$$

and we already know that the tangent of the initial angle θ is the slope of the initial velocity vector. We can also see this by evaluating the derivative of the trajectory Eq(1) at the origin (initial slope), which gives

$$\frac{dy}{dx}(x = 0) = \tan(\theta)$$

so the slopes are related as the negative reciprocal, which is the condition for being perpendicular. QED

To complete this, we use the x-tangent expression in Eq(2) to find the y-coordinate of the tangent point:

$$\boxed{Y_{tangent} = \frac{v_0^{\,2}}{2\,g}\left(1 - \frac{1}{\tan(\theta)^2}\right) + y_0}$$

In the first figure above one such point is illustrated by a box at the tangent for a trajectory.