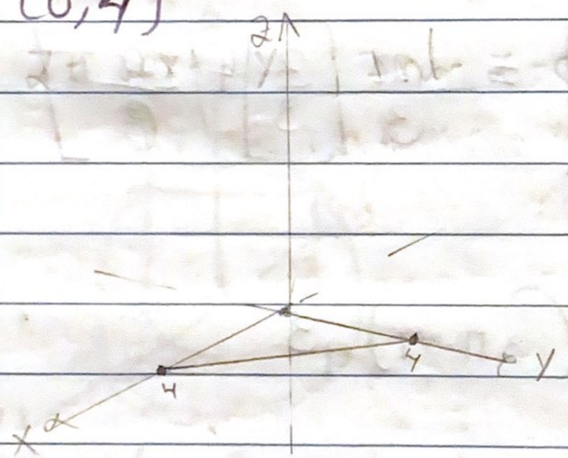
Ejercicios 14.5

Áreas de la superficie

Hallar el área de la superficie dada por $z = f(x,y)$ sobre la región R. (Sugerencia: Algunas de las integrales son más sencillas en Coordenadas Polares).

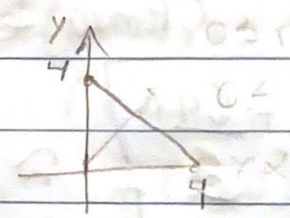
1) $f(x,y) = 2x + 2y$

R: triángulo cuyos vértices son $(0,0)$, $(4,0)$, $(0,4)$



$f_x = 2$

$f_y = 2$



$$A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \int_0^4 \int_0^{4-x} \sqrt{1 + (2)^2 + (2)^2}\, dy\, dx = \int_0^4 \int_0^{4-x} \sqrt{9}\, dy\, dx$$

$$= \int_0^{4-x} 3\, dy = 3\left[y\right]_0^{4-x} = 12 - 3x$$

$$\int_0^4 12 - 3x\, dx = 12x - \frac{3}{2}x^2 \Big|_0^4 = 12(4) - \frac{3}{2}(4)^2 = 48 - 24$$

$$= 24 \text{ unid}^2$$

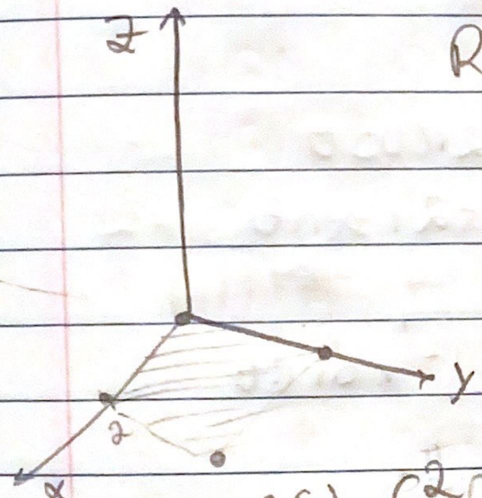
5) $f(x,y) = 9 - x^2$

R: Cuadrado cuyos vertices son $(0,0)$, $(2,0)$, $(0,2)$, $(2,2)$

$$A(S) = \iint_D \sqrt{1+f_x^2+f_y^2}\,dA$$

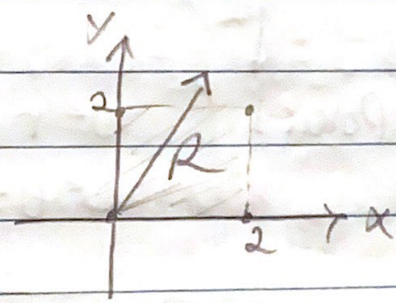
$f_x = -2x$

$f_y = 0$



R: $0 \le X \le 2$

$0 \le x \le 2$



$$A(S) = \int_0^2\int_0^2 \sqrt{1+(-2x)^2+(0)^2}\,dy\,dx$$

$$\sqrt{1+4x^2}\,dy\,dx$$

$$= \int_0^2 \sqrt{1+4x^2}\,dy = \sqrt{1+4x^2}\,\Big[y\Big]_0^2 = 2\sqrt{1+4x^2}$$

$$\int_0^2 2\sqrt{1+4x^2}\,dx = \frac{2}{2}\Big[\sqrt{1+4x^2}\cdot 2x + \ln\left(\sqrt{1+4x^2}+2x\right)\Big]_0^2$$

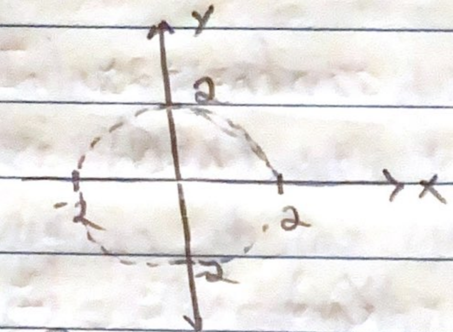
$$= \left(\sqrt{1+4(2)^2}\cdot 2(2) + \ln\left(\sqrt{1+4(2)^2}+2(2)\right) - \ln\left(\sqrt{1+4(0)^2}+2(0)\right)\right)$$

$$= 4\sqrt{17} + \ln(\sqrt{17}+4) - \ln(1)$$

$$= 4\sqrt{17} + \ln(\sqrt{17}+4) \text{ unidades}^2$$

10) $f(x,y) = 13 + x^2 - y^2$

$R = \{(x,y) : x^2 + y^2 \leq 4\}$



$R: (x,y) : x^2 + y^2 \leq 4$

$f_x = 2x$

$f_y = -2y$

R: (Polar) $0 \leq r \leq 2$ $f_x = 2r\cos\theta$

$0 \leq \theta \leq 2\pi$ $f_y = -2r\,\mathrm{sen}\,\theta$

$$A(S) = \int_0^{2\pi}\int_0^2 \sqrt{1 + (2r\cos t)^2 + (-2r\,\mathrm{sen}\,\theta)^2}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2\cos^2\theta + 4r^2\mathrm{sen}^2\theta}\; r\,dr\,d\theta$$

$$= \int_0^2 \sqrt{1 + 4r^2}\; r\,dr\,d\theta =$$

$u = \sqrt{1+4r^2}$

$u^2 = 1 + 4r^2$

$2u\,du = 8r\,dr$

$\frac{1}{4}u\,du = r\,dr$

$$\int_1^9 \frac{1}{4}\, u \cdot u\, du = \frac{1}{4}\int_1^9 u^2\, du$$

$$= \frac{1}{4}\left[\frac{u^3}{3}\right]_1^9 = \frac{1}{12}\left[\sqrt{1+4r^2}^{\,3}\right]_0^2$$

$$= \frac{1}{12}\left[\left(\sqrt{1+4(2)^2}\right)^3 - \left(\sqrt{1+4(0)^2}\right)^3\right]$$

$$= \frac{1}{12}\left(\left(\sqrt{17}\right)^3 - (1)^3\right) = \frac{1}{12}\left(\sqrt{17}^{\,3} - 1\right)$$

Continuacion=

$$\int_0^{2\pi}\left(\frac{1}{12}\left(\sqrt{17}^3-1\right)\right)d\theta$$

$$= \frac{1}{12}\left(\sqrt{17}^3-1\right)\left[\theta\right]_0^{2\pi}$$

$$= \frac{1}{12}\left(\sqrt{17}^3-1\right)\left(2\pi\right)$$

$$= \frac{\left(\left(\sqrt{17}\right)^3-1\right)\pi}{6} \text{ unid}^2$$

$$\frac{\left(17\sqrt{17}-1\right)\pi}{6} \text{ unid}^2$$