PROJECTILE MOTION *Equations*

- **General time of flight (TOF)**

$$T = \frac{1}{g}\left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2\, g\, y_0}\right]$$

- **General range**

$$R = v_0 \cos(\theta)\, T \qquad R = \frac{v_0}{g} \cos(\theta)\left[v_0 \sin(\theta) + \sqrt{(v_0 \sin(\theta))^2 + 2\, g\, y_0}\right]$$

- **Angle for maximum range ("optimum angle")**

$$\theta_{opt} = \frac{\pi}{2} - \operatorname{atan}\left(\sqrt{1 + \frac{2\, g\, y_0}{v_0^2}}\right) = \frac{\pi}{2} - \operatorname{atan}\left(\frac{v_f}{v_0}\right) = \operatorname{atan}\left(\frac{v_0}{v_f}\right)$$

$$\theta_{opt} = \operatorname{atan}\left(\frac{v_0}{\sqrt{v_0^2 + 2\, g\, y_0}}\right) \qquad \operatorname{asin}\left[\frac{1}{\sqrt{2\left(1 + \frac{g\, y_0}{v_0^2}\right)}}\right] \qquad \operatorname{acos}\left(\frac{\sqrt{2}}{2}\sqrt{\frac{1 + \frac{2\, g\, y_0}{v_0^2}}{1 + \frac{g\, y_0}{v_0^2}}}\right)$$

- **Maximum range (i.e. range at optimum angle)**

$$R_{max} = \frac{v_0}{g}\sqrt{v_0^2 + 2\, g\, y_0} = \frac{v_0\, v_f}{g}$$

- **TOF at optimum angle**

$$\frac{1}{g}\sqrt{2\left(v_0^2 + g\, y_0\right)}$$

- **Maximum y at optimum angle**

$$y_0 + \frac{v_0^4}{4\, g\left(v_0^2 + g\, y_0\right)}$$

- **Final angle for optimum initial angle (max range)**

$$\theta_T = \operatorname{atan}\left(\frac{\sqrt{v_0^2 + 2\,g\,y_0}}{v_0}\right) \qquad \theta_{opt} + \theta_T = \frac{\pi}{2}$$

also, using negative of this angle for final angle, then the initial slope (tangent of initial angle) and final slopes are negative inverses, so vectors are perpendicular

$$x_{intersect} = \frac{y_0 + \tan(\theta_T)\,R_{max}}{\tan(\theta_T) - \tan(\theta_{opt})} \qquad y_{intersect} = \tan(\theta_{opt})\,x_{intersect} + y_0$$

$$x_{intersect} = \frac{v_0}{2\,g}\sqrt{v_0^2 + 2\,g\,y_0} = \frac{R_{max}}{2} \qquad y_{intersect} = y_0 + \frac{v_0^2}{2\,g}$$

point of intersection of these tangent lines (use negative for theta-T); y is on directrix

$$y_T = \tan(\theta_T)\,(x - R_{max}) \qquad y = \tan(\theta_{opt})\,x + y_0$$

equations of tangent lines

- **Special cases**

zero initial height note max R here is half the max height (i.e., at 90 degrees)

$$T := \frac{2\,v_0}{g}\sin(\theta) \qquad R := \frac{2\,v_0^2}{g}\sin(\theta)\cos(\theta) = \frac{v_0^2}{g}\sin(2\,\theta)$$

$$\theta_{maxR} := \frac{\pi}{2} - \operatorname{atan}(1) = \frac{\pi}{4} \qquad R_{max} = \frac{v_0^2}{g} \qquad \theta_X := \frac{1}{2}\operatorname{asin}\left(\frac{X\,g}{v_0^2}\right)$$

zero angle

$$R := v_0\sqrt{\frac{2\,y_0}{g}} \qquad T := \sqrt{\frac{2\,y_0}{g}}$$

- **Equations of motion: parametric** usually x(0) = 0 and accel in x is 0

$$x(t) := x_0 + v_0\,t\cos(\theta) + \frac{1}{2}a_x\,t^2 \qquad v_x(t) := v_0\cos(\theta)$$

$$y(t) := y_0 + v_0\,t\sin(\theta) - \frac{1}{2}g\,t^2 \qquad v_y(t) := v_0\sin(\theta) - g\,t$$

- **Equations of motion: y(x)** as v0 becomes large, this becomes linear (frozen rope)

$$y(x) = \left[\frac{-g}{2\,(v_0\cos(\theta))^2}\right]x^2 + \tan(\theta)\,x + y_0 \qquad \frac{dy}{dx} = \frac{-g}{(v_0\cos(\theta))^2}\,x + \tan(\theta)$$

- **Vertex form of parabola**

$$x_{maxY} := \frac{v_0^{\,2}}{g}\sin(\theta)\cos(\theta) \qquad y_{maxY} := y_0 + \frac{v_0^{\,2}}{2\,g}\sin(\theta)^2 \qquad t_{maxY} := \frac{v_0}{g}\sin(\theta)$$

$$y(x) = \frac{-g}{2\left(v_0\cos(\theta)\right)^2}\left(x - \frac{v_0^{\,2}}{2\,g}\sin(2\,\theta)\right)^2 + \left(y_0 + \frac{v_0^{\,2}}{2\,g}\sin(\theta)^2\right) \qquad |p| = \frac{\left(v_0\cos(\theta)\right)^2}{2\,g}$$

focus

$$y_D = y_0 + \frac{v_0^{\,2}}{2\,g} \quad \text{directrix} \qquad x_F = \frac{v_0^{\,2}}{2\,g}\sin(2\,\theta) \qquad y_F = y_0 + \frac{v_0^{\,2}}{2\,g}\left(1 - 2\cos(\theta)^2\right)$$

zero initial height, optimum angle (45 degrees)

$$x_{maxY} = \frac{v_0^{\,2}}{2\,g} \qquad y_{maxY} = \frac{v_0^{\,2}}{4\,g} \qquad t_{maxY} = \frac{v_0}{g\sqrt{2}}$$

- **Velocity vector**

$$v(t) := \sqrt{g^2\left(t - \frac{v_0\sin(\theta)}{g}\right)^2 + \left(v_0\cos(\theta)\right)^2} \qquad \textbf{magnitude}$$

$$\alpha(t) := \operatorname{atan}\left(\tan(\theta) - \frac{g\,t}{v_0\cos(\theta)}\right) \qquad \textbf{angle vs. time}$$

$$\alpha(x) = \operatorname{atan}\left[\frac{-g\,x}{\left(v_0\cos(\theta)\right)^2} + \tan(\theta)\right] \qquad \textbf{angle vs. x}$$

- **Final (impact) velocity**

$$\left|v_T\right| := \sqrt{v_0^{\,2} + 2\,g\,y_0} \qquad \textbf{magnitude}$$

$$\alpha(T) := \operatorname{atan}\left[-\sqrt{\tan(\theta)^2 + \frac{2\,g\,y_0}{\left(v_0\cos(\theta)\right)^2}}\right] \qquad \textbf{angle}$$

$$y_{tangent}(x) := (x - R)\left[-\sqrt{\tan(\theta)^2 + \frac{2\,g\,y_0}{\left(v_0\cos(\theta)\right)^2}}\right] \qquad \textbf{tangent line equation}$$

$$v_y(T) := \sqrt{\left(v_0\sin(\theta)\right)^2 + 2\,g\,y_0} \qquad \textbf{y component}$$

- **Galileo angles (for equal range; zero initial y only)**

$$\theta_1 := \frac{\pi}{4} - \frac{1}{2}\,\mathrm{acos}\left(\frac{g\,R}{{v_0}^2}\right) \qquad \theta_2 := \frac{\pi}{4} + \frac{1}{2}\,\mathrm{acos}\left(\frac{g\,R}{{v_0}^2}\right) \qquad \theta_1 + \theta_2 := \frac{\pi}{2} \qquad \frac{T_1}{T_2} := \tan(\theta_1)$$

- **Maxima ellipse** all the vertices (maxima) of a set of trajectories for given initial velocity, and zero initial y, as angle varies to 90 degrees, will fall on this ellipse

$$x^2 + 4\left(y - \frac{{v_0}^2}{4\,g}\right)^2 = \frac{{v_0}^4}{4\,g^2}$$

- **Targeting solutions** given initial velocity and height (can be nonzero), and a point (X,Y), find the angle to hit the point

$$\beta = \frac{-g}{2\,{v_0}^2} \qquad a = \beta\,X^2 \qquad b = X \qquad c = y_0 - Y + \beta\,X^2 \qquad D = b^2 - 4\,a\,c$$

$D > 0$ two angles $D < 0$ no angles $D = 0$ one angle

$$\psi_1 = \frac{-b + \sqrt{D}}{2\,a} \qquad \psi_2 = \frac{-b - \sqrt{D}}{2\,a} \qquad \theta_1 = \mathrm{atan}(\psi_1) \qquad \theta_2 = \mathrm{atan}(\psi_2)$$

parabola of safety $Y = \frac{-g}{2\,{v_0}^2}\,X^2 + y_0 + \frac{{v_0}^2}{2\,g}$ target outside this envelope cannot be hit, regardless of angle

$X_{tangent} = \frac{{v_0}^2}{g\,\tan(\theta)}$ $Y_{tangent} = \frac{{v_0}^2}{2\,g}\left(1 - \tan(\theta)^{-2}\right) + y_0$ point where trajectory is tangent to envelope

Velocity vector slope (i.e., derivative of envelope) at tangent point $\frac{-1}{\tan(\theta)}$ Velocity initial slope $\tan(\theta)$

so these are perpendicular

$T = \frac{X}{v_0\,\cos(\theta)}$ time of flight to target

$|v(T)| = \sqrt{{v_0}^2 - 2\,g\,X\,\tan(\theta) + \left(\frac{g\,X}{v_0\,\cos(\theta)}\right)^2}$ velocity magnitude at target

$\alpha(T) = \mathrm{atan}\left[\tan(\theta) - \frac{g\,X}{(v_0\,\cos(\theta))^2}\right]$ velocity angle at target

- **First-order air resistance (drag) solution**

$$x = \frac{v_0 \cos(\theta)}{k} (1 - \exp(-k\,t)) \qquad y = \frac{1}{k}\left(\frac{g}{k} + v_0 \sin(\theta)\right)(1 - \exp(-k\,t)) - \frac{g\,t}{k} + y_0$$

$$v_x(t) = v_0 \cos(\theta) \exp(-k\,t) \qquad v_y(t) = \frac{-g}{k} + \left(\frac{g}{k} + v_0 \sin(\theta)\right)\exp(-k\,t)$$

$$Tmax = \frac{-1}{k} \ln\left(\frac{1}{1 + \frac{v_0\,k}{g}\sin(\theta)}\right)$$

k as used here depends on mass; *smaller* for larger mass, *smaller* for smaller drag coeff; note larger mass approaches vacuum solution (all these equations do)

$$x_{max} = \frac{v_0^2 \sin(\theta)\cos(\theta)}{g + v_0 \sin(\theta)\,k} \qquad y_{max} = y_0 + \frac{v_0 \sin(\theta)}{k} + \frac{g}{k^2}\ln\left(\frac{1}{1 + \frac{v_0\,k}{g}\sin(\theta)}\right)$$

$$T = \frac{1}{g + k\,v_0 \sin(\theta)}\left[v_0 \sin(\theta) + \sqrt{v_0 \sin(\theta)\left(v_0 \sin(\theta) + 2\,k\,y_0\right) + 2\,g\,y_0}\right]$$

this TOF is approximate, only good for small k, approaches non-drag for small k; since this is not a very good approximation, no point in doing range, etc.

- **Tilted terrain** ϕ is the angle of the terrain, positive counterclockwise from horizontal at y0; bogus solutions are possible with these; use carefully; watch negative x,y

$$x_{Intersection} = \frac{2\left(v_0 \cos(\theta)\right)^2}{g}\left(\tan(\theta) - \tan(\phi)\right) \qquad y_{Intersection} = \tan(\phi)\,x_I + y_0$$

$$L = \frac{x_I}{\cos(\phi)}$$ distance along hill

vertex coordinates same as for flat terrain ($\phi = 0$)

$$T = \frac{v_0}{g}\left[\sin(\theta) + \sqrt{\sin(\theta)^2 - 4\tan(\phi)\left[\cos(\theta)\left(\sin(\theta) - \tan(\phi)\cos(\theta)\right)\right]}\right]$$

flat terrain, same as y0 = 0 above for TOF

$$\theta_{opt} = \frac{\pi}{4} + \frac{\phi}{2}$$ launch angle for max downslope distance, which is $$L_{opt} = \frac{v_0^2}{g}\,\frac{1}{1 + \sin(\phi)}$$

- **Line of sight angle** measured positive counterclockwise from launch point; target at (X,Y)

$$\phi = \operatorname{atan}\left(\frac{Y - y_0}{X}\right)$$ if Y = 0, X = R

$$\phi = \operatorname{atan}\left(-\sqrt{\frac{g\,y_0}{2\,v_0^2}}\right)$$ for $\theta = 0$; airplane / bomb