MAA [1.4] GEOMETRIC SEQUENCES

SOLUTIONS

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O. Practice questions

1. (a) $u_1 = 10,\ \ r = 2$

(b) 5120

(c) 10230

(d) $10 \times 2^{n-1}$ $(= 5 \times 2^{n})$

(e) $n = 12$

2. (a) $u_1 = 10,\ \ r = 0.5$

(b) 0.0195

(c) 19.98

(d) $10 \times 0.5^{n-1}$ $(= 20 \times 0.5^{n})$

(e) $n = 6$

(f) Since $-1 < r < 1$, $S_\infty = \dfrac{10}{1-0.5} = 20$

3. (a) $r = 2$

(b) 0.0195

(c) 80

(d) 9

(e) $u_9 = 1280$

(f) Since $r > 1$

4. (i) $S_\infty = \dfrac{1}{1-\dfrac{2}{5}} = \dfrac{5}{3}$ (ii) $S_\infty = \dfrac{1}{1+\dfrac{2}{5}} = \dfrac{5}{7}$

5. (a) (i) $2k - k = k + 60 - 2k \Leftrightarrow 2k = 60 \Leftrightarrow k = 30$

(ii) 30,60,90 and $d = 30$

(b) (i) $\dfrac{2k}{k} = \dfrac{k+60}{2k} \Leftrightarrow 2 = \dfrac{k+60}{2k} \Leftrightarrow 4k = k + 60 \Leftrightarrow 3k = 60 \Leftrightarrow k = 20$

(ii) 20,40,80 and $r = 2$

6. (i) 2046 (ii) 1

7. (a) $5 \times 3^{n-1} < 100000$, $n = 10$

(b) $u_{10} = 98415$

(c) $\dfrac{5 \times (3^{n} - 1)}{3-1} < 100000$, $n = 9$

8. (a) $r=3,\ u_1=5$

(b) $r=\frac{1}{3},\ u_1=71,744,535$

(c) $r=3,\ u_1=5$ OR $r=-3,\ u_1=-5$

(d) $r=\frac{1}{2},\ u_1=20$

A. Exam style questions (SHORT)

9. (a) $u_{10}=3(0.9)^9$

(b) $r=0.9$

$S=\frac{3}{1-0.9}=\frac{3}{0.1}=30$

10. (a) $\frac{1}{5}$ (0.2)

(b) (i) $u_{10}=25\left(\frac{1}{5}\right)^9=0.0000128$, (ii) $u_n=25\left(\frac{1}{5}\right)^{n-1}$

(c) $S=\frac{125}{4}=31.25\ (=31.3 \text{ to } 3\,s\,f)$

11. (a) $r=\frac{2}{3}$

(b) $u_{15}=1.39$

(c) $S=1215$

12. (a) $\frac{54}{18}=\frac{162}{54}=\frac{486}{162}\ (=3)$ hence geometric

(b) (i) $r=3,\quad u_n=18\times 3^{n-1}$

(ii) $18\times 3^{n-1}=1062882 \Leftrightarrow n=11$

13. (a) $\frac{a}{8}=\frac{1}{2} \Leftrightarrow a=4$ **OR** $\frac{2}{a}=\frac{1}{2} \Leftrightarrow a=4$

(b) $8\left(\frac{1}{2}\right)^7=0.0625$

(c) $\dfrac{8\left(\left(\frac{1}{2}\right)^{12}-1\right)}{\frac{1}{2}-1}=16.0(3\ s.f)\ \ (=4095/256)$

14. (a) 0.5 $\left(\frac{1}{2}\right)$

(b) (i) $a=4$ (ii) $b=1$

(c) $\frac{16(1-0.5^n)}{(1-0.5)}=31.9375$

$n=9$

15. (a) dividing two terms *e.g.* $-\frac{1800}{3000}, -\frac{1800}{1080}$

$r = -0.6$

(b) $u_{10} = 3000(-0.6)^9 = -30.2$ (exact value -30.233088)

(c) $S = \frac{3000}{1.6} = 1875$

16. (a) $r = \frac{16}{32}\left(= \frac{1}{2}\right)$

(b) $u_6 = 32 \times \left(\frac{1}{2}\right)^{6-1} = 1$

OR 32, 16, 8, 4, 2, 1 $u_6 = 1$

(c) $S_\infty = \frac{32}{1-\frac{1}{2}} = 64$

17. $r = -\frac{1}{3}$ $S_\infty = \frac{27}{1+\frac{1}{3}}$ $S_\infty = \frac{81}{4}$ $(=20.25)$

18. $S = \frac{u_1}{1-r} = \frac{\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5}$

19. $u_1 = -12$ and $r = \frac{-2}{3}$

$S_\infty = \frac{-12}{1-\left(-\frac{2}{3}\right)} = \frac{-36}{5}$ or -7.2

20. (a) $u_1 = 48, = u_2 = 192$,

$r = \frac{u_2}{u_1} = \frac{192}{48} = 4$

(b) $S_n = \frac{u_1(r^n - 1)}{(r-1)} = \frac{48(4^n - 1)}{3} = 16(4^n - 1)$

21. $2 \times 1.05^{n-1} > 500$ so $1.05^{n-1} > 250$

METHOD A: Trial and error;

The smallest integer that satisfies the inequality is $n = 115$. Then $u_{115} = 521$

METHOD B: By using GDC SolveN or Graphical solution

The smallest integer that satisfies the inequality is $n = 115$. Then $u_{115} = 521$

22. (a) $\sum_{n=1}^{3}\left(3 \times 2^n\right) = 6 + 12 + 24 = 42$

(b) $\sum_{n=1}^{12}\left(3 \times 2^n\right) = 24570.$

23. (a) $r = \frac{2500}{2000} = 1.25$

(b) $S_6 = \frac{2000(1.25^6 - 1)}{1.25 - 1} = 22517.57813........ = 22518$ (to the nearest dollar)

24. (a) $r = \frac{8320}{8000} \Leftrightarrow r = 1.04$

(b) Fees = $8000\ (1.04)^6 = 10122.55$ USD (USD not required)

(c) Total = $\frac{8000(1.04^8 - 1)}{1.04 - 1} = 73713.81$ USD (USD not required)

25. (a) (i) 2 minutes + 6 seconds + 6 seconds = 2 minutes 12 seconds (or 2.2 minutes)

(ii) $2(1.05)^2 = 2.205$

(b) $S_{10} = \frac{2(1.05^{10} - 1)}{(1.05 - 1)} = 25.2$ minutes (or 25 minutes 12 seconds)

(c) the common difference for John is 6 seconds = 0.1 minutes

$S_{10} = \frac{10}{2}(2 \times 2 + 9 \times 0.1) = 24.5$ minutes

(or 24 minutes 30 seconds)

26. (a) Let the population at the end of 1999 be x.

$\frac{44100}{x} = \frac{x}{40000} \Leftrightarrow x = 42\ 000$

(b) $r = \frac{44100}{42000} = 1.05$

$u_n = u_1 r^{n-1}$

METHOD A

Assume that u_1 if for 1992 and $u_5 = 40\ 000$ is for 1996

$40\ 000 = u_1(1.05)^4$

$u_1 = 32\ 908$ (or 32 900 to 3 s.f.)

METHOD B

For 4 years before 1996 we **divide** 40 000 by $(1.05)^4$

$\frac{40000}{(1.05)^4} = 32908$

27. (a) $28 = 7r^2$
$r = 2$

(b) 114681

28. (a) $u_3 = 8 \Leftrightarrow 8 = 18r^2 \Leftrightarrow r^2 = \frac{8}{18}\left(=\frac{4}{9}\right) \Leftrightarrow r = \pm\frac{2}{3}$

(b) $S_\infty = \frac{u_1}{1-r}$,

$S_\infty = 54, \frac{54}{5}(=10.8)$ $S_\infty = 54$ and $S_\infty = \frac{54}{5}$ (=10.8)

(c) 18, 12, 8 and 18, –12, 8

29.

(a) $u_1 r^3 = -\frac{2}{3}$

$\Rightarrow 18r^3 = -\frac{2}{3}$

$\Rightarrow r^3 = -\frac{1}{27} \Rightarrow r = -\frac{1}{3}$

(b) $S_\infty = \frac{u_1}{1-r} = \frac{18}{1+\frac{1}{3}} = \frac{27}{2}$

30. (a) $r = \frac{36}{108}\left(\frac{1}{3}\right)$

(b) $u_1\left(\frac{1}{3}\right)^7 = 36 \Leftrightarrow$ $u_1 = 78732$

(c) $118096 = \frac{78732\left(1-\left(\frac{1}{3}\right)^k\right)}{\left(1-\frac{1}{3}\right)}$

$k = 10$

31. (a) $u_1 r^4 = 324 \Leftrightarrow u_1 r = 12 \Leftrightarrow r^3 = 27 \Leftrightarrow r = 3$

(b) $4 \times 3^9 = 78732$

(c) $4 \times 3^{k-1} > 2000$

$k > 6$, So $k = 7$

32. (a) $u_4 = u_1 r^3 \Leftrightarrow \frac{1}{81} r^3 = \frac{1}{3} \Leftrightarrow r = 3$

(b) $\frac{\frac{1}{81}(3^n - 1)}{2} > 40; \Leftrightarrow n > 7.9888...$ So $n = 8$

(c) $\mathbf{S_7} = 13.49...$;, $\mathbf{S_8} = 40.49...$ which is > 40

33. (a) (i) –2/3 (ii) –243 (b) –133 (c) -729/5

34. (a) $\frac{x}{5} = \frac{45}{x} \Leftrightarrow x^2 = 225 \Leftrightarrow x = 15$ or $x = -15$

(b) if $x = 15$, then $y = 135$, if $x = -15$, then $y = -135$

35. $1 - a = b - 1$ and $b = a^2 \Leftrightarrow a^2 + a - 2 = 0 \Leftrightarrow a = -2, b = 4$

36. $k + \frac{2}{3}k + \left(\frac{2}{3}\right)^2 k + \left(\frac{2}{3}\right)^3 k + ... = 1$

$k\left(\frac{1}{1-\frac{2}{3}}\right) = 1$ so $k = \frac{1}{3}$

37. (a) $-1 < \frac{2x}{3} < 1$. This gives $-1.5 < x < 1.5$ or $|x| < \frac{3}{2}$

(b) When $x = 1.2$, the common ratio is $r = 0.8$ and the sum is $\frac{1}{1-0.8} = 5$

38. (a) $r = 4 - 3x$

(b) $-1 < 4 - 3x < 1 \Rightarrow 1 < x < \frac{5}{3}$

(c) $x = 1.2 \Rightarrow u_1 = 0.8$ $r = 0.4$

(i) $S_\infty = \frac{0.8}{0.6} = \frac{4}{3}$ (=1.333…)

(ii) $S_n = \frac{0.8(1-0.4^n)}{0.6} > 1.328$, Solving gives $n > 6.02$, 7 terms are needed.

39. (a) $\frac{u_1(1-r^2)}{1-r} = 15$ (1), $\frac{u_1}{1-r} = 27$ (2)

Divide (1) by (2): $1 - r^2 = \frac{15}{27} = \frac{5}{9} \Rightarrow r^2 = \frac{4}{9} \Rightarrow r = \frac{2}{3}$

(b) $u_1 = 27 \times \frac{1}{3} = 9$

40. $\frac{u_1}{1-r} = \frac{27}{2}$ (1) and $\frac{u_1(1-r^3)}{1-r} = 13$ (2)

Divide (1) by (2): $1 - r^3 = \frac{13}{\frac{27}{2}} = \frac{26}{27} \Rightarrow r^3 = \frac{1}{27} \Rightarrow r = \frac{1}{3}$

Therefore, $u_1 = 9$.

41. $S_\infty = \frac{u_1}{1-r} = 32$ and $S_4 = \frac{u_1(1-r^4)}{1-r} = 30$

Divide S_4 by S_∞: $1 - r^4 = \frac{30}{32} = \frac{15}{16} \Leftrightarrow r^4 = \frac{1}{16} \Leftrightarrow r = \frac{1}{2}$ and so $u_1 = 16$

$S_\infty - S_8 = 32 - \frac{16(1-0.5^8)}{1-0.5} = 32 - 32(1-0.5^8) = 32 \times 0.5^8 = 0.0125$

42. (a) $u_{11} = u_1 + 10d \Leftrightarrow -16 + 10d = 39 \Leftrightarrow 10d = 55 \Leftrightarrow d = 5.5$

(b) $u_3 = u_1 r^2 \Leftrightarrow u_1 r^2 = 12$

$u_5 = u_1 r^4 \Leftrightarrow u_1 r^4 = \frac{16}{3}$

$$r^2 = \frac{\left(\frac{16}{3}\right)}{12} = \frac{16}{36} = \frac{4}{9} \Leftrightarrow r = \frac{2}{3}$$

43. (a) $u_{96} = u_1 + 95d = 0 + 95 \times 12 = 1140$

(b) $6r^5 = 16d \Leftrightarrow 6r^5 = 16 \times 12 \Leftrightarrow 6r^5 = 192 \Leftrightarrow r^5 = 32 \Leftrightarrow r = 2$

(c) $0 + (n-1) \times 12 = 6 \times 2^{n-1} \Leftrightarrow n = 2$ or $n = 3$

(Indeed, the 2nd term of each sequence is 12, the 3rd term of each sequence is 24)

44. $u_1 = 2 (= S_1) \qquad u_2 = S_2 - u_1 = 10 \Rightarrow d = 8$

$u_{32} = 250$

u_2, u_m and u_{32} in geometric progression $\Rightarrow \frac{u_m}{u_2} = \frac{u_{32}}{u_m}$

$\Rightarrow u_m^2 = u_2 \times u_{32} = 10 \times 250 \Rightarrow u_m = 50$

($u_m = -50$ not possible) $u_m = 50 = 2 + 8(m-1) \Rightarrow m = 7$.

45. (a) let the first three terms of the geometric sequence be

$u_1 + 2d$, $u_1 + 3d$ and $u_1 + 6d$

$$\frac{u_1 + 6d}{u_1 + 3d} = \frac{u_1 + 3d}{u_1 + 2d}$$

$u_1^2 + 8u_1 d + 12d^2 = u_1^2 + 6u_1 d + 9d^2$

$2u_1 + 3d = 0$

$u_1 = -\frac{3}{2}d$ *or* $a = -\frac{3}{2}d$

(b) $$r = \frac{a + 6d}{a + 3d} = \frac{-1.5d + 6d}{-1.5d + 3d} = \frac{4.5d}{1.5d} = 3$$

46. (a) $$r = \frac{u_1 + 15d}{u_1 + 10d} = \frac{u_1 + 10d}{u_1}$$

$u_1^2 + 20u_1 d + 100d^2 = u_1^2 + 15u_1 d$

$\Rightarrow 0 = 5u_1 d + 100d^2 \Rightarrow 0 = u_1 + 20d \;\; u_1 = -20d$

$$r = \frac{u_1 + 10d}{u_1} \left(= \frac{-20d + 10d}{-20d}\right) \Rightarrow r = \frac{1}{2}$$

(b) $$18 = \frac{u_1}{1 - \frac{1}{2}} \Rightarrow u_1 = 9$$

$d = -\frac{9}{20}$ (= -0.45)

B. Exam style questions (LONG)

47. (a) 0.5 (b) 31.25 (c) 999.023 (d) 7 (e) 7.8125

48. (a) $r = \frac{360}{240} = \frac{240}{160} = \frac{3}{2} = 1.5$

(b) 2002 is the 13^{th} year.
$u_{13} = 160(1.5)^{13-1} = 20759$

(c) $5000 = 160(1.5)^{n-1} \Rightarrow n = 9.49 \Rightarrow 10^{\text{th}}$ year $\Rightarrow$ 1999

OR

Find , $u_9 = 4100.625$ $u_{10} = 6150.9375. \Rightarrow 10^{\text{th}}$ year $\Rightarrow$ 1999

(d) $S_{13} = 160\left[\frac{1.5^{13}-1}{1.5-1}\right] = 61958$

(e) Nearly everyone would have bought a portable telephone so there would be fewer people left wanting to buy one **OR** Sales would saturate.

49. (a) (i) Area B $= \frac{1}{16}$, area C = 64

(ii) $\frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{4}$ $\frac{\frac{1}{64}}{\frac{1}{16}} = \frac{1}{4}$ (Ratio is the same.)

(iii) Common ratio $= \frac{1}{4}$

(b) (i) Total area (S_2) $= \frac{1}{4} + \frac{1}{16} = \frac{5}{16}$ = (= 0.3125) (0.313, 3 s.f.)

(ii) Required area $= S_8 = \frac{\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^8\right)}{1-\frac{1}{4}}$ = 0.333328 2(471...) = 0.333328 (6 s.f.)

(c) Sum to infinity $= \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$

50. (a) (i) $\text{PQ} = \sqrt{\text{AP}^2 + \text{AQ}^2} = \sqrt{2^2+2^2} = \sqrt{4(2)} = 2\sqrt{2}$ cm

(ii) Area of PQRS = $(2\sqrt{2})$ $(2\sqrt{2})$ = 8 cm^2

(b) (i) Side of third square $= \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{4} = 2$ cm
Area of third square = 4 cm^2

(ii) $\frac{1^{\text{st}}}{2^{\text{nd}}} = \frac{16}{8}$ $\frac{2^{\text{nd}}}{3^{\text{rd}}} = \frac{8}{4} \Rightarrow$ Geometric progression, $r = \frac{8}{16} = \frac{4}{8} = \frac{1}{2}$

(c) (i) $u_{11} = u_1 r^{10} = 16\left(\frac{1}{2}\right)^{10} = \frac{16}{1024} = \frac{1}{64}$ (= 0.015625 = 0.0156, 3 s.f.)

(ii) $S_\infty = \frac{u_1}{1-r} = \frac{16}{1-\frac{1}{2}} = 32$

51. (a) (i) $r = -2$

(ii) $u_{15} = -3\,(-2)^{14} = -49152$ (accept -49200)

(b) (i) 2, 6, 18

(ii) $r = 3$

(c) $\frac{x+1}{x-3} = \frac{2x+8}{x+1} \Leftrightarrow x^2 + 2x + 1 = 2x^2 + 2x - 24 \Leftrightarrow x^2 = 25 \Leftrightarrow x = 5$ or $x = -5$

$x = -5$

(d) (i) $r = \frac{1}{2}$

(ii) $S = \frac{-8}{1-\frac{1}{2}}$ $S = -16$

52. (a) $\frac{1-3^n}{1-3} = 29\,524 \Leftrightarrow n = 10.$

(b) Common ratio is $\frac{1}{3}$, $(0.333\,(3s.f.)$

(c) $\frac{1-\left(\frac{1}{3}\right)^{10}}{1-\frac{1}{3}} = 1.50$ (3s.f.)

(d) Both $\left(\frac{1}{3}\right)^{10}$ and $\left(\frac{1}{3}\right)^{1000}$ (or those numbers divided by 2/3) are 0 when corrected to 3s.f., so they make no difference to the final answer.

Notes: Accept any valid explanation

(e) The sequence given is $G_1 + G_2$

The sum is 29 524 + 1.50 = 29 525.5

53. (a) $1024r^3 = 128 \Leftrightarrow r^3 = \frac{1}{8} \Leftrightarrow r = \frac{1}{2} = 0.5$

(b) $1024 \times 0.5^{10} = 1$

(c) $S_8 = \frac{1024\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}} = 2040$

(d) $\frac{1024\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} > 2047.968$

$n = 16$

(e) $S_{15} = 2047.9375$

$S_{16} = 2047.96875$

So $n = 16$

54. (a) (i) T_1 = \$250
d = \$200
$T_{10} = 250 + 9 \times 200 = 2050$

(ii) T_1 = \$10
$r = 2$
$T_{10} = 10 \times 2^9 = 5120$

(b) $S_{10} = \frac{10}{2}(250 + 2050) = 11500$ **OR** $S_{10} = \frac{10}{2}\{2 \times 250 + (10 - 1) \times 200\} = 11500$

(c) Option One: \$10000
Option Two: \$11500
Option Three: $S_{10} = \frac{10(2^{10} - 1)}{2 - 1} = 10\,230$

Therefore, Option Two would be best.

55. (a) $u_1 = 135 + 7(1) = 142$

(b) $u_2 = 135 + 7(2) = 149$
$d = 149 - 142 = 7$ (**OR** *alternatives*)

(c) $S_n = \frac{n[2(142) + 7(n-1)]}{2} = \frac{n[277 + 7n]}{2} = \frac{7n^2}{2} + \frac{277n}{2}$ $(= 3.5n^2 + 138.5n)$

(d) $20r^3 = 67.5 \Leftrightarrow r^3 = 3.375 \Leftrightarrow r = 1.5$

(e) $T_7 = \frac{20(1.5^7 - 1)}{(1.5 - 1)} = 643$ *(accept 643.4375)*

(f) $\frac{20(1.5^n - 1)}{(1.5 - 1)} > \frac{7n^2}{2} + \frac{277n}{2}$

$n = 10$