[MAA 2.4-2.5] COMPOSITION – INVERSE FUNCTION

SOLUTIONS

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O. Practice questions

1. (a) $(f \circ g)(x) = 10 - 10x$, $(g \circ f)(x) = 50 - 10x$

(b) $f^{-1}(x) = \dfrac{10 - x}{2}$

(c) $g^{-1}(10) = 2$

(d) $(f^{-1} \circ g)(x) = \dfrac{10 - 5x}{2}$, $(g \circ f)^{-1}(x) = \dfrac{50 - x}{10}$

(e) $(f \circ f)(x) = 4x - 10$ $(g \circ g)(x) = 25x$

2.

Original function	Inverse function
$f(x) = x + 5$	$f^{-1}(x) = x - 5$
$f(x) = x - 5$	$f^{-1}(x) = x + 5$
$f(x) = x + 100$	$f^{-1}(x) = x - 100$
$f(x) = 3x$	$f^{-1}(x) = x/3$
$f(x) = x/5$	$f^{-1}(x) = 5x$
$f(x) = x^3$	$f^{-1}(x) = \sqrt[3]{x}$
$f(x) = 3x + 100$	$f^{-1}(x) = \dfrac{x - 100}{3}$

3. (a) (i) $f(1) = 3$ (ii) $f^{-1}(1) = 5$

(b) $x = 6$

(c) $x = 4$

4. (a) $g(3) = 1$ $f^{-1}(3) = 4$

(b) $(f \circ g)(2) = -1$

(c) $(g \circ g)(3) = 5$

(d) $x = 1$

5. $\dfrac{x}{x+5} = y \Leftrightarrow x = xy + 5y \Leftrightarrow x(1-y) = 5y \Leftrightarrow x = \dfrac{5y}{1-y}$

$f^{-1}(x) = \dfrac{5x}{1-x}$ (or $\dfrac{-5x}{x-1}$)

6. **METHOD A**

(a) $(g \circ f)(x) = \dfrac{2^x}{2^x - 2}$. Hence $(g \circ f)(3) = \dfrac{2^3}{2^3 - 2} = \dfrac{8}{6} = \dfrac{4}{3}$

(b) $\dfrac{x}{x-2} = y \Rightarrow y(x-2) = x \Rightarrow yx - 2y = x \Rightarrow x(y-1) = 2y \Rightarrow x = \dfrac{2y}{y-1}$

$g^{-1}(x) = \frac{2x}{x-1}$

Hence $g^{-1}(5) = \frac{10}{(5-1)} = 2.5$

METHOD B

(a) $f(3) = 8, \ (g \circ f)(3) = g(8) = \frac{8}{6} = \frac{4}{3}$

(b) $\frac{x}{x-2} = 5 \Leftrightarrow x = 5x - 10 \Leftrightarrow 4x = 10 \Leftrightarrow x = \frac{5}{2}$

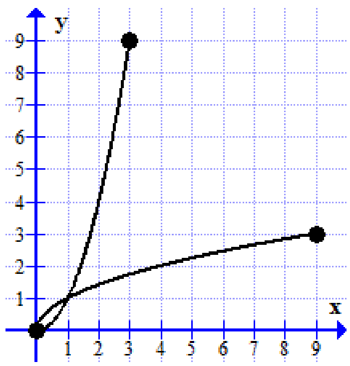
Hence $g^{-1}(5) = \frac{5}{2} = 2.5$

7. (a) *parabola between* $x = 0$ and $x = 3$

(b) $0 \le x \le 3 \quad 0 \le y \le 9$

(c) $y = \sqrt{x}$

(d)



graph symmetric about the line $y = x$

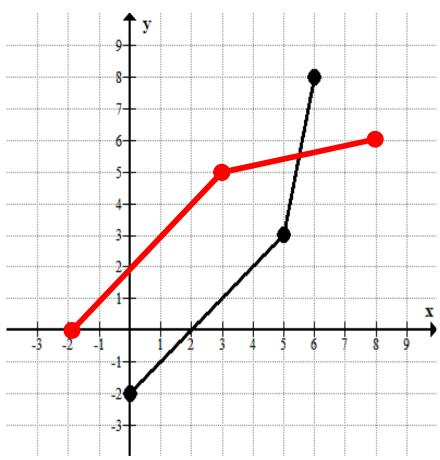
(e) $0 \le x \le 9 \quad 0 \le y \le 3$

8. (a) (i) $f(0) = -2$ (ii) $f(2) = 0$ (iii) $f(4) = 2$

(b) (i) $f^{-1}(3) = 5$ (ii) $f^{-1}(8) = 6$ (iii) $f^{-1}(-1) = 1$ (iv) $f^{-1}(0) = 2$

(c) $x = 2$

(d) $x = 4$



9. (a) $\frac{2x-3}{3x-2} = y \Leftrightarrow 3xy - 2y = 2x - 3 \Leftrightarrow 3xy - 2x = 2y - 3$

$\Leftrightarrow x(3y-2) = 2y - 3 \Leftrightarrow x = \frac{2y-3}{3y-2}$

Hence $f^{-1}(x) = \dfrac{2x-3}{3x-2} = f(x)$

(b) $(f \circ f)(x) = \dfrac{2\dfrac{2x-3}{3x-2} - 3}{3\dfrac{2x-3}{3x-2} - 2} = \dfrac{2(2x-3) - 3(3x-2)}{3(2x-3) - 2(3x-2)} = \dfrac{-5x}{-5} = x$

10. (a) $f^{-1}(x) = \sqrt[3]{x}$

(b) $f(g(x)) = x + 1 \Rightarrow [g(x)]^3 = x + 1$

so $g(x) = \sqrt[3]{x+1}$

OR $g = f^{-1} \circ (f \circ g)$, so $g(x) = \sqrt[3]{x+1}$

(b) $g(f(x)) = x + 1 \Rightarrow g(x^3) = x + 1$

so $g(x) = \sqrt[3]{x} + 1$

OR $g = (g \circ f) \circ f^{-1}$, so $g(x) = \sqrt[3]{x} + 1$

11. (a) $f = h \circ g^{-1}$

(b) $g = f^{-1} \circ h$

(c) $g = f^{-1} \circ k \circ h^{-1}$

A. Exam style questions (SHORT)

12. (a) $(h \circ g)(x) = \dfrac{5(3x-2)}{(3x-2)-4} = \dfrac{15x-10}{3x-6}$

(b) numerator = 0 ⇒ M1) $x = \dfrac{2}{3}$ $(= 0.667)$

13. (a) $(f \circ g): x \mapsto 3(x + 2)$ $(= 3x + 6)$

(b) **METHOD 1**

$f^{-1}(x) = \dfrac{x}{3}$ $g^{-1}(x) = x - 2$

$f^{-1}(18) = \dfrac{18}{3} = 6$ $g^{-1}(18) = 18 - 2 = 16$

$f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22$

METHOD 2

$3x = 18, x + 2 = 18$

$x = 6, x = 16$

$f^{-1}(18) + g^{-1}(18) = 6 + 16 = 22$

14. (a) $y = 2x + 1 \Rightarrow x = \dfrac{y-1}{2}$, $f^{-1}(x) = \dfrac{x-1}{2}$

(b) $g(f(-2)) = g(-3) = 3(-3)^2 - 4 = 23$

(c) $f\left(g(x)\right) = f(3x^2 - 4) = 2(3x^2 - 4) + 1 = 6x^2 - 7$

15. $\sqrt{3-2x} = 5 \Leftrightarrow 3 - 2x = 25 \Leftrightarrow -2x = 22 \Leftrightarrow x = -11$

OR

Let $y = \sqrt{3-2x} \Rightarrow y^2 = 3 - 2x \Rightarrow x = \dfrac{3-y^2}{2}$

$\Rightarrow f^{-1}(x) = \dfrac{3-x^2}{2} \Rightarrow f^{-1}(5) = \dfrac{3-25}{2} = -11$

16. (a) **METHOD 1**

$f(3) = \sqrt{7}$ $\quad (g \circ f)(3) = 7$

METHOD 2

$(g \circ f)(x) = \sqrt{x+4}^{\,2} = x + 4$

$(g \circ f)(3) = 7$

(b) $y = \sqrt{x+4} \Rightarrow y^2 = x + 4 \Rightarrow x = y^2 - 4$

$f^{-1}(x) = x^2 - 4$

(c) $x \geq 0$

17. (a) **METHOD 1**

$f(-2) = -12$

$(g \circ f)(-2) = g(-12) = -24$

METHOD 2

$(g \circ f)(x) = 2x^3 - 8$

$(g \circ f)(-2) = -24$

(b) $y = x^3 - 4 \Rightarrow y + 4 = x^3 \Rightarrow x = \sqrt[3]{y+4}$

$f^{-1}(x) = \sqrt[3]{(x+4)}$

18. (a) **METHOD 1**

$(f \circ g)(4) = f(g(4)) = f(1) = 2$

METHOD 2

$(f \circ g)(x) = \dfrac{2}{x-3}$ $\quad$ so $\quad (f \circ g)(4) = 2$

(b) Let $y = \dfrac{1}{x-3} \Rightarrow y(x-3) = 1 \Rightarrow x - 3 = \dfrac{1}{y} \Rightarrow x = \dfrac{1}{y} + 3 \ \left(= \dfrac{1+3y}{y}\right)$

$g^{-1}(x) = \dfrac{1}{x} + 3 \ \left(= \dfrac{1+3x}{x}\right)$

(c) $x \neq 0$ (or $\mathbb{R} \setminus \{0\}$)

19. (a) For $f^{-1}(2)$, $3x + 5 = 2 \Rightarrow x = -1$

$f^{-1}(2) = -1$

(b) $g(f(-4) = g(-12 + 5) = g(-7) = 2(1 + 7) = 16$

20. (a) $(g \circ f)(x) = 2\cos x + 1$

(b) $(f \circ g)(x) = 2\cos(2x + 1)$

(c) $(g^{-1} \circ g)(x) = x$, $(g \circ g^{-1})(x) = x$

21. (a) $y = \frac{6-x}{2} \Rightarrow x = 6 - 2y \Rightarrow g^{-1}(x) = 6 - 2x$

(b) $(f \circ g^{-1})(x) = 4[(6 - 2x) - 1] = 4(5 - 2x) = 20 - 8x$

$20 - 8x = 4 \Leftrightarrow 8x = 16 \Leftrightarrow x = 2$

22. (a) $y = 2x - 3 \Rightarrow 2x = y + 3 \Rightarrow x = \frac{y+3}{2}$ $\quad g^{-1}(x) = \frac{x+3}{2}$

(b) **METHOD 1**

$g(4) = 5 \Rightarrow f(5) = 25$

METHOD 2

$f \circ g(x) = (2x - 3)^2$

$f \circ g(4) = (2 \times 4 - 3)^2 = 25$

23. (a) $(g \circ f)(x) = 7 - 2x + 3 = 10 - 2x$

(b) $g^{-1}(x) = x - 3$

(c) **METHOD 1**

$g^{-1}(5) = 2$

$f(2) = 3$

METHOD 2

$(f \circ g^{-1})(x) = 7 - 2(x - 3) = 13 - 2x$

$(f \circ g^{-1})(5) = 3$

24. (a) $h(x) = f(2x - 5) = 6x - 15$

(b) $6x - 15 = y \Rightarrow 6x = y + 15 \Rightarrow x = \frac{y+15}{6}$

$h^{-1}(x) = \frac{x+15}{6}$

25. (a) $f^{-1}(x) = x^2$

$(f^{-1} \circ g)(x) = f^{-1}(2^x) = 2^{2x}$

(b) $2^{2x} = 16 \Rightarrow 2x = 4 \Rightarrow x = 2$

26. $f^{-1}(x) = \frac{x+5}{3}$ $\quad g^{-1}(x) = x + 2$

$(f^{-1} \circ g)(x) = \frac{x+3}{3}$ $\quad (g^{-1} \circ f)(x) = 3x - 3 \Leftrightarrow$

$\frac{x+3}{3} = 3x - 3 \Leftrightarrow x + 3 = 9x - 9 \Leftrightarrow x = \frac{12}{8} = \frac{3}{2}$

27. (a) $h(x) = g(f(x)) = \dfrac{4}{x+2} - 1$

(b) $y = \dfrac{4}{x+2} - 1 \Rightarrow y+1 = \dfrac{4}{x+2} \Rightarrow x+2 = \dfrac{4}{y+1} \Rightarrow x = \dfrac{4}{y+1} - 2$

Hence $h^{-1}(x) = \dfrac{4}{x+1} - 2$ (or $\dfrac{2-2x}{x+1}$)

28. $(f \circ g) : x \mapsto x^3 + 1$

$(f \circ g)^{-1} : x \mapsto (x - 1)^{1/3}$

29. (a) $g(x) = (f \circ f)(x) = \dfrac{\frac{x}{x+1}}{\frac{x}{x+1} + 1} = \dfrac{x}{2x+1}$

(b) $(g \circ g)(2) = g\left(\dfrac{2}{5}\right) = \dfrac{2}{9}$

30. (a)

$y = \dfrac{3x-4}{x+2}$

$xy + 2y = 3x - 4$

simplifying $x(y-3) = -2y - 4$

expressing y in terms of x, $x = \dfrac{2y+4}{3-y}$

interchanging x and y, $y = \dfrac{2x+4}{3-x}$

$f^{-1}(x) = \dfrac{2x+4}{3-x}$

(b) Domain $x \neq 3$

31. (a) $f(x) = (x-3)^2 + 4 = x^2 - 6x + 9 + 4 = x^2 - 6x + 13$

(b) $y = (x-3)^2 + 4$

$y - 4 = (x-3)^2$

$\sqrt{y-4} = x - 3$

$\sqrt{y-4} + 3 = x$

$\Rightarrow f^{-1}(x) = \sqrt{x-4} + 3$

(c) $x \geq 3$ and $y \geq 4$

(d) $x \geq 4$ and $y \geq 3$

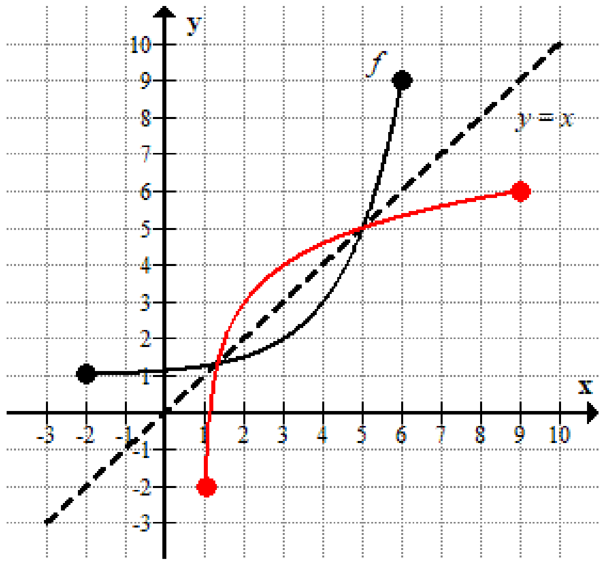
32. (a) $(f \circ f)(1) = f(f(1)) = f(3) = 2$

(b) $(g^{-1} \circ f)(4) = g^{-1}(f(4)) = g^{-1}(1) = 3$

(c) $f(g(x)) = 1$, so $g(x) = 4$, so $x = 1$

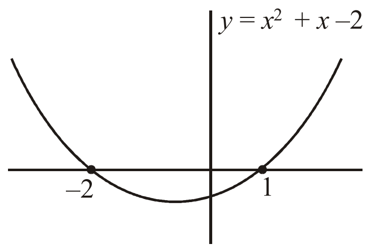
(d) $(g^{-1} \circ g)(2) = 2$

33.



34. (a) $(f \circ g)(x) = \sqrt{x^2 + x - 2}$

$x^2 + x - 2 \geq 0$



$a = -2,\ b = 1$

(b) range is $y \geq 0$

35. (a) in both cases it is the x-coordinate of the vertex.

EITHER by the formula $x = -\dfrac{b}{2a}$ **OR** by GDC graph, minimum

(i) $a = 2$, (ii) $b = 2$

(b) **EITHER by solving** $f(x) = 7 \Leftrightarrow 3x^2 - 12x + 7 = 7$ **OR** by GDC graph x-CAL

(i) $f^{-1}(7) = 4$, (ii) $g^{-1}(7) = 0$

36. **METHOD A**

We firstly find $f^{-1}(x) = \sqrt[3]{x+1}$

(a) $g = f^{-1} \circ (f \circ g)$, so $g(x) = \sqrt[3]{2x+2}$

(b) $g = (g \circ f) \circ f^{-1}$, so $g(x) = 2\sqrt[3]{x+1} + 1$

METHOD B

(a) $f(g(x)) = x + 1 \Rightarrow [g(x)]^3 - 1 = 2x + 1$, so $g(x) = \sqrt[3]{2x+2}$

(b) $g(f(x)) = 2x + 1 \Rightarrow g(x^3 - 1) = 2x + 1$

Set $y = x^3 - 1$, then $x = \sqrt[3]{y+1}$, so $g(x) = 2\sqrt[3]{x+1} + 1$

37.

$$g^{-1}(x) = \frac{x+1}{2}$$

$$f(x) = f \circ g \circ g^{-1}(x) = \frac{\frac{x+1}{2}+1}{2}$$

$$= \frac{x+3}{4}$$

$$f(x-3) = \frac{(x-3)+3}{4}$$

$$= \frac{x}{4}$$

38. Let $y = \dfrac{x^2-1}{x^2+1} => yx^2 + y = x^2 - 1$

$x^2(1-y) = 1 + y => x^2 = \dfrac{1+y}{1-y} => x = \pm\sqrt{\dfrac{1+y}{1-y}}$

Interchanging, $f^{-1}(x) = -\sqrt{\dfrac{1+x}{1-x}}$

39. (a) By using GDC, graph, x-CAL **OR** SolveN

(i) $f^{-1}(1) = 0$, (ii) $f^{-1}(0) = -0.682$, (iii) $f^{-1}(2) = 0.682$.

(b) Since $f(x) = f^{-1}(x) = x$, we solve instead, $f(x) = x$

EITHER by graph, **OR** by SolveN **OR** analytically $x^3 + x + 1 = x \Leftrightarrow x^3 = -1$

The solution is $x = -1$

B. Exam style questions (LONG)

40. (a) $(f \circ f)(0) = f(1) = -5$

(b) $f(x) = 2(x-2)^2 - 7$

(c) $y \geq -7$

(d) $f^{-1}(x) = 2 - \sqrt{\dfrac{x+7}{2}}$

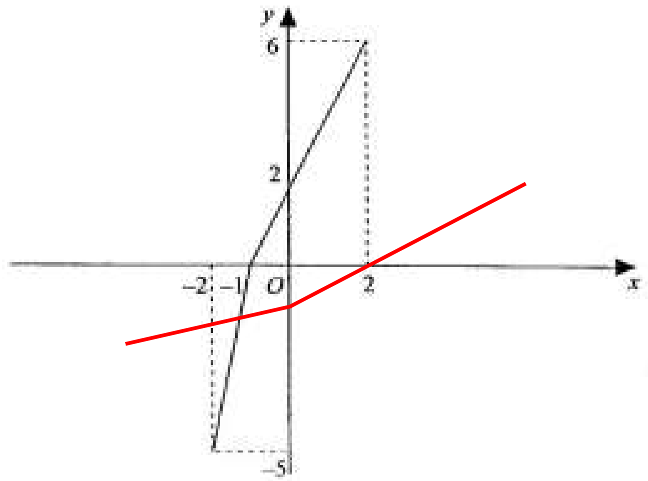
(e) Domain: $x \geq -7$, Range: $y \leq 2$

41. (a) We show that $f^{-1}(x) = \dfrac{3x-1}{x-3}$

(b) $(f \circ f)(k) = k$

(c) $(f \circ g)(-2) = f(-5) = \dfrac{-16}{-8} = 2$.

(d) $-5 \le x \le 6$.



42.

(a) $f\ g(x) = (ax+b+2)^2 - 3$

$= a^2x^2 + abx + 2ax + abx + b^2 + 2b + 2ax + 2b + 4 - 3$

$= a^2x^2 + x(2ab+4a) + (b^2+4b+1)$

$\Rightarrow a^2x^2 + x(2ab+4a) + (b^2+4b+1) \equiv 4x^2 + 6x - \dfrac{3}{4}$

Equating coefficients of x^2 gives $a^2 = 4$

$\Rightarrow a = 2$, since $a > 0$

Equating coefficients of x gives $2ab + 4a = 6$

$\Rightarrow 2a(b+2) = 6$

$\Rightarrow 4(b+2) = 6$

$\Rightarrow b = -\dfrac{1}{2}$

(b) $h\ k(x) = 5(cx^2 - x + 2) + 2 = 0$

$\Rightarrow 5cx^2 - 5x + 10 + 2 = 0$

$\Rightarrow 5cx^2 - 5x + 12 = 0$

Condition for real equal roots is: $b^2 - 4ac = 0$

$\Rightarrow 25 - 240c = 0$

$\Rightarrow c = \dfrac{5}{48}$

43. (a) $x = -1$

(b) $f^{-1}(x) = (x-1)^2 - 1 = x^2 - 2x$

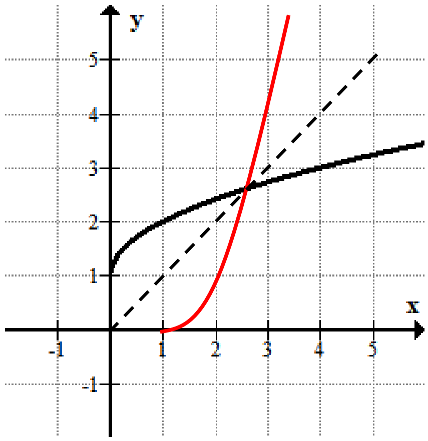
(c) $h(x) = (x^2 - 2x)^2$

(d) $k(x) = x^4 - 2x^2$

44. (a) $f^{-1}(x) = (x-1)^2$.

(b) $x \geq 1,\ y \geq 0$

(c)



(d) We solve $f(x) = x$ **OR** $f^{-1}(x) = x$

$$x = \frac{3+\sqrt{5}}{2}.$$