SOME INTEGRAL-CALCULUS FACTOIDS

The basic rule for definite integrals, such as the ones we need for "work" calculations, is

$$\int_a^b f(x)\,dx \;=\; F(x)\Big|_a^b \;=\; F(b) - F(a)$$

where F is the "antiderivative" of *f(x)*. Some of these are simple enough to remember, or become automatic with repeated use, but the best bet is to look them up in tables. The basic ones are given on the AP formula sheet. An example:

$$\int_0^\pi \sin(\theta)\,d\theta = -\cos(\theta)\Big|_0^\pi = -\big[(-1) - (+1)\big] = \cos(\theta)\Big|_\pi^0 = 1 - (-1) = 2$$

A negative sign for the integral can be evaluated by reversing the upper and lower limits; this is sometimes easier, as here.

If there are quantities in the integrand that are not functions of the integration variable, those quantities can be taken outside the integral:

$$\int_1^2 \mu m g\; x^3 dx \;=\; \mu m g \int_1^2 x^3\,dx = \frac{\mu m g}{4}\,x^4\Big|_1^2 = \frac{\mu m g}{4}\left(2^4 - 1^4\right) \;=\; 3.75\,\mu m g$$

The mean or average value of a function over a finite range of the independent variable can be found using the "mean-value theorem of integral calculus"

$$\bar{y} \;=\; \frac{1}{b-a}\int_a^b f(x)\,dx$$

A more general definition is, with *w(x)* a "weight,"

$$\bar{y} \;=\; \frac{\int_a^b f(x)\,w(x)\,dx}{\int_a^b w(x)\,dx}$$

For example, find the average value of $f(x) = x^2$ over $0 \le x \le 2$.

$$\bar{y} \;=\; \frac{1}{2-0}\int_0^2 x^2\,dx \;=\; \frac{1}{2}\,\frac{x^3}{3}\bigg|_0^2 \;=\; \frac{1}{6}\left(2^3 - 0^3\right) \;=\; \frac{4}{3}$$

Note that the TI calculators have a function for numerical integration. For this example it would be *fnInt(x^2, x, 0, 2)*. This is useful for checking results.

INTEGRALS OF MOTION: WORK

A force is applied to an object for a finite time and over a finite displacement. This leads to a finite change in the velocity of the object. Note that $F = ma$ by itself has no bounds and would result in an undefined acceleration and hence velocity. Thus we should integrate F over space or time. In one dimension, for clarity, this is

$$\int_a^b F(x)\,dx \;\Rightarrow\; \Delta E_K = \frac{1}{2}m\left(v_b^2 - v_a^2\right) \tag{1}$$

or

$$\int_{t1}^{t2} \mathbf{F}(t)\,dt \;\Rightarrow\; \Delta\mathbf{p} = \frac{1}{2}\,m\left(\mathbf{v}_{t2} - \mathbf{v}_{t1}\right) \tag{2}$$

For the moment, all we need to get out of these integrals is that a finite change in velocity is obtained when we apply a force for a finite time or across a finite displacement.

Integrating across displacement, Eq(1), results in a scalar quantity called "work." In the most general case it is found using the *line integral* over some finite portion of a space (3D) curve C:

$$W = \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r} \tag{3}$$

but we will not need to evaluate these vector integrals in this course. In this case the force may vary in magnitude and/or direction with position along the curve, and those curves can have all sorts of complicated 3D shapes. Note that the integrand is a dot product, and the result of this integration is a scalar, a single number. In some problems, Eq(3) can be adapted to find a solution without the complete line-integral formalism.

Many problems are in one dimension, and we can write a simpler but still useful version of the work integral:

$$W = \int_a^b F(x)\,\cos(\theta_x)\,dx \tag{4}$$

Here both the magnitude of the force and/or its direction θ_x with respect to the single dimension x may vary along the path. As long as this variation can be defined mathematically, and is "integrable," then we can find the work done in applying this nonconstant force over $a \le x \le b$. An example of this situation is pushing a lawnmower in one pass across a level yard. As the mower is pushed, both the "amount" or "strength" of push and the angle of push might vary across the path.

Another variation on Eq(3) is the case of a constant force. Then we don't need the integral and can write

$$W = \mathbf{F} \bullet \Delta\mathbf{r} \tag{5}$$

This form is useful when the force is given in *(i,j,k)* format, as is the displacement. Note that the displacement $\Delta\mathbf{r}$ is now a finite difference, not a differential, as it was in Eq(3). For this equation we would use the component-multiplication method for the dot product evaluation.

An even simpler version, in one dimension with a constant force, is

$$W = F\,\Delta x\,\cos(\theta) \tag{6}$$

and if the angle between the force and displacement is zero we have the standard "force times distance" result

$$W = F\,\Delta x \tag{7}$$

WORK-ENERGY: PART I

Having calculated this quantity "work" we might wonder of what use it is. The work integral, again in one dimension for clarity, is, with the understanding that the force F is the component along the displacement,

$$W = \int_A^B F(x)\, dx$$

but, using Newton's Second Law, this can also be written as

$$W = \int_A^B m\, a\, dx \;=\; m\int_A^B \frac{dv}{dt}\, dx$$

With a change of the variable of integration from x to the velocity v, we have

$$W \;=\; m\int_A^B \frac{dv}{dt}\, dx \;=\; m\int_{v(A)}^{v(B)} v\, dv \;=\; \frac{m}{2}\left(v_B^2 - v_A^2\right) \;\equiv\; \Delta E_K$$

where the quantity ½ mv^2 is defined to be the **kinetic energy** E_K of the object. There are several ways to effect this change of variable; the most common is the chain rule:

$$\frac{dv}{dt} \;=\; \frac{dv}{dx}\,\frac{dx}{dt} \;=\; v\,\frac{dv}{dx}$$

Another way to look at it is to use the kinematic relation

$$v_t^2 \;=\; v_0^2 \;+\; 2\,a\left(x_t - x_0\right)$$

and then differentiate this implicitly with respect to x (not t). That will give

$$2\,v\,\frac{dv}{dx} = 2\,a \;\;\Rightarrow\;\; m\int_A^B a\, dx \;=\; m\int_{v(A)}^{v(B)} v\,\frac{dv}{dx}\, dx$$

which leads to the same result for the change in kinetic energy that we found above. This math is only needed in the derivation of the **work-energy theorem**; in practice all we need to know is

$$\boxed{W = \Delta E_K}$$

This says that a given amount of work done on an object will result in a certain change in the object's kinetic energy.

Both work and energy have units of joules (J), which are N-m, kg-m^2/s^2. This very simple relation allows easy solutions for certain types of mechanics problems, particularly those that involve velocities. The work W can be found using

$$W = \int_C \mathbf{F}(\mathbf{r}) \bullet d\mathbf{r}$$

or any of its simpler variations. In some cases, a solution using Newton's Laws and kinematics would be extremely tedious, and perhaps not tractable at all, but using work-energy can lead to a straightforward solution.

The concepts of work and energy do not appear in Newton's *Principia*. These were later developments, and there was scholarly disagreement about these ideas. It is the case in other areas of physics besides mechanics, e.g., thermodynamics, that the concept of "work" is usefully thought of as a *transfer of energy*. In mechanics that transfer of energy is via the application of a force. However, we have not defined "energy." That term sometimes is defined as "the capacity to do work." So we see that these definitions are a bit circular. Nonetheless, the concepts of work and energy are, without question, very useful.