How to calculate mathematics π(圆周率的计算方法)

From 翔文公益数学

Taylor's theorem and Taylor series

In this section, we shall consider a polynomial approximation which mimics a function f near one given point. We will seek coefficients $a_0, a_1, \cdots, a_n$ such that the polynomial

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + R((x-a)^{n+1})$$

approximates to $f(x)$ near $x = a$, where assuming that $f(x)$ is nth differentiable at $x = a$. Therefore $f(x) \approx P_n(x)$, and we try to choose the coefficients $a_n, n = 0, 1, 2, \cdots, n$.

The Taylor series of a real or complex-valued function $f(x)$ that is infinitely differentiable at a real or complex number a is the power series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

where $n!$ denotes the factorial of n and $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a. In the more compact sigma notation, this can be written as

$$泰勒级数 f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The derivative of order zero of f is defined to be f itself and $(x-a)^0$ and $0!$ are both defined to be 1. When $a = 0$, the series is also called a [Maclaurin series][2].

For instance:

The Taylor series for the exponential function e^x at $a = 0$ is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x^n}{n!} &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots.\end{aligned}$$

The above expansion holds because the derivative of e^x with respect to x is also e^x and e^0 equals 1. This leaves the terms $(x-0)^n$ in the numerator and $n!$ in the denominator for each term in the infinite sum.

Euler formula: $e^{ix} = \cos(x) + \mathrm{i}\ \sin(x)$

The Taylor series of function e^z at $a = 0$ is

$$\begin{aligned}e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots \\ &= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} + \cdots .\end{aligned}$$

Let $z = ix$, where i is imaginary unit, which satisfied $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, i^{4k+3} = -i$.

left side means $e^{ix} = \cos(x) + i\ \sin(x)$

right side equals to $1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots .$

Therefore

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The above sums are the taylor series sum for function $cos(x)$ and $sin(x)$. (they have the following **Colin Maclaurin series**, for all x)

$$\begin{aligned}\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .\end{aligned}$$

$$\begin{aligned}\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .\end{aligned}$$

可见，$\cos(x)$ **是偶函数，**$\sin(x)$ **是奇函数。**

Of course, we can use Taylor expansion to get the above formula.

$$\sin^{(2k)}(x) = \frac{d^{(2k)}\sin(x)}{dx} = (-1)^k \sin(x)$$

$$\sin^{(2k+1)}(x) = (-1)^k \cos(x)$$

The even-th derivative function of $sin(x)$ at $x = 0$ equals to 0, and odd-th derivative function of $sin(x)$ at $x = 0$ equals to $(-1)^k$

Polynomial approximation theorem

Question: How to solve the quation $sin(x) = 0$?

1. First, $\sin(x) = 0$ has the solution $\{k\pi,\ \ k = 0, \pm 1, \pm 2, \cdots .\}$

2. According to the fundamental theorem of algebra (polynomial approximation theorem) , Let's suppose $\sin(x)$ is a polynomial.

$$\sin(x) = c * \prod_{k=1}^{\infty}(k\pi - x)(k\pi + x)x$$

$$\frac{\sin(x)}{x} = c * \prod_{k=1}^{\infty}(k\pi - x)(k\pi + x)$$

Now limit at x tends to 0, we shall get

$$c = \prod_{k=1}^{\infty}\frac{1}{k^2\pi^2}$$

Therefore we got

$$\begin{aligned}\sin(x) &= x\prod_{k=1}^{\infty}(1 - \frac{x}{k\pi})(1 + \frac{x}{k\pi}) \\ &= x\prod_{k=1}^{\infty}[1 - \frac{x^2}{(k\pi)^2}]\end{aligned}$$

Let $x = \frac{\pi}{2}$, we get

$$\begin{aligned}\frac{\pi}{2} &= \prod_{k=1}^{\infty}\frac{2k}{2k-1}\cdot\frac{2k}{2k+1} \\ &= \prod_{k=1}^{\infty}\frac{1}{1-\frac{1}{4k^2}} \\ &= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots \\ &= \frac{4}{3}\cdot\frac{16}{15}\cdot\frac{36}{35}\cdot\frac{64}{63}\cdots\end{aligned}$$

see also John Wallis' product for π

Similarly, we can get

$$\begin{aligned}\cos(x) &= \prod_{k=1}^{\infty}(1 - \frac{2x}{(2k-1)\pi})(1 + \frac{2x}{(2k-1)\pi}) \\ &= x\prod_{k=1}^{\infty}(1 - \frac{4x^2}{(2k-1)^2\pi^2})\end{aligned}$$

$$\tan(x) = \frac{x}{\frac{\pi}{4}}\cdot\prod_{k=1}^{\infty}\frac{1-\frac{x}{k\pi}}{1-\frac{1}{4k}}\cdot\frac{1+\frac{x}{k\pi}}{1+\frac{1}{4k}}$$

Euler identity

1. From Taylor expansion, we get
   $$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
2. From polynomial approximation theorem and equation solution, we get
   $$\sin(x) = x\prod_{k=1}^{\infty}(1 - \frac{x^2}{(k\pi)^2})$$
3. Compare the degree 3 of x-term, we get the coefficients shoulb be equivalent.
   $$\frac{1}{3!} = \sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}$$
   Therefore
   $$\begin{aligned}\frac{\pi^2}{6} &= \sum_{k=1}^{\infty}\frac{1}{k^2}\\ &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \cdots\end{aligned}$$
   收敛速度还可以接受。

Gregory-Leibniz series

The series for the **inverse tangent** function, which is also known as **Gregory's series**, can be given by:

$$\begin{aligned}\arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\\ &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}\end{aligned}$$

推导过程如下:

$$\because \frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}$$
$$\because 1 - x^n = (1-x)(1 + x + x^2 + \cdots + x^{n-1})$$
$$\therefore \frac{1}{1-x} = \sum_{n=0}^{\infty}x^n,\ |x| < 1$$
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$
$$\begin{aligned}\therefore \arctan(x) &= \int\frac{1}{1+x^2}\mathrm{d}x\\ &= \int\sum_{n=0}^{\infty}(-1)^n x^{2n}\mathrm{d}x\\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},\ |x| < 1\end{aligned}$$

The **Leibniz formula** for $\frac{\pi}{4}$ can be obtained by putting $x = 1$ $(TODO?)$ into the above inverse-tangent series.

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

As you can see, this converges very slowly(**收敛极慢！不可取**), with large, alternating over-estimates and under-estimates.

Nilakantha Series

$$\begin{aligned}\pi &= 3 + \frac{4}{2\cdot3\cdot4} - \frac{4}{4\cdot5\cdot6} + \frac{4}{6\cdot7\cdot8} - \frac{4}{8\cdot9\cdot10} + \cdots \\ &= 3 + \frac{1}{1\cdot3\cdot2} - \frac{1}{2\cdot5\cdot3} + \frac{1}{3\cdot7\cdot4} - \frac{1}{4\cdot9\cdot5} + \cdots \\ &= 3 + \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)n(2n-1)}\end{aligned}$$

This is the **faster convergent method** for pi.
这是计算圆周率的更快一点的收敛方法。

Viete's Formula

Viète's formula is the following infinite product of nested radicals representing the mathematical constant π:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

It is named after François Viète (1540–1603), who published it in 1593.

Viète's formula may be rewritten and understood as a limit expression.

$$\lim_{n\to\infty} \prod_{i=1}^{n} \frac{a_i}{2} = \frac{2}{\pi}$$

where $a_n = \sqrt{2 + a_{n-1}}$, with initial condition $a_1 = \sqrt{2}$.

Viète's formula may be obtained as a special case of a formula given more than a century later by Leonhard Euler, who discovered that:

$$\frac{\sin x}{x} = \cos\frac{x}{2} \cdot \cos\frac{x}{4} \cdot \cos\frac{x}{8} \cdots$$

Substituting
$x = \frac{\pi}{2}$ in this formula yields:

$$\frac{2}{\pi} = \cos\frac{\pi}{4} \cdot \cos\frac{\pi}{8} \cdot \cos\frac{\pi}{16} \cdots$$

Then, expressing each term of the product on the right as a function of earlier terms using the half-angle formula:

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$$

gives Viète's formula.

It is also possible to derive from Viète's formula a related formula for π that still involves nested square roots of two, but uses only one multiplication: ref Viete formula

$$\pi = \lim_{k\to\infty} 2^k \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}}_{k\text{ square roots}}$$

反三角函数 $\arcsin(x)$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned}\implies \sin^{-1}x &= \int (1-x^2)^{-1/2}\mathrm{d}x + c(constant)\\ &= \int (1+\frac{x^2}{2}+\frac{3x^4}{8}+\frac{5x^6}{16}+\cdots+c),\ |x|<1\\ &= x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots+c\end{aligned}$$

$$\because \sin^{-1}x = 0 at x = 0 \implies c = 0$$

$$\therefore \sin^{-1}(x) = x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots,\ |x|<1$$
$$= \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1}$$

同理可以得到

$$\therefore \cos^{-1}(x) = \frac{\pi}{2} - \left(x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots\right),\ |x|<1$$
$$= \frac{\pi}{2} - \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}x^{2n+1}$$

Taylor级数和Maclaurin Series的实现

在GeoGebra中可以轻松实现多项式之和逼近各种函数。

1. 先定义次数Order的滑条 $n = slider(1,20,1)$
2. 再定义要逼近的函数，如 $f(x) = \arctan(x), g(x) = \sin(x), h(x) = \cos(x), \cdots$
3. 调用函数 $g = TaylorPolinomial(f, x(A), n),\ A = (0,0)$
4. 调用函数 $FormulaText(g, true, true)$ 可以显示级数

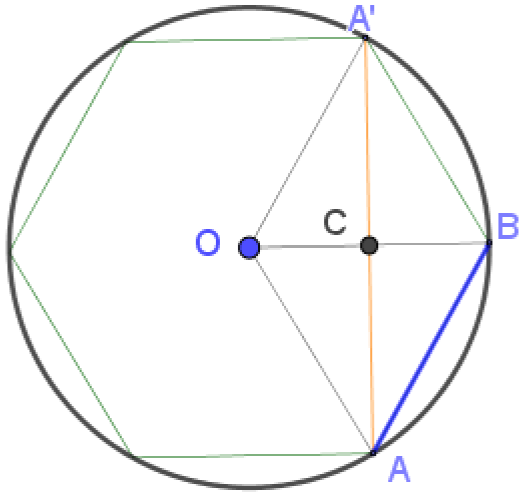
TaylorSeries的多项式逼近演示GGB

计算 π 收敛速度极快算法

代数和几何结合的方法。

采用刘徽割圆术，用正n边形逼近圆的方法，实现计算圆周率 π 的目的。

记 正n边形的边长为 L_n, 由正n边形产生的正 2n 边形的边长为 L_{2n}, 则如图可有



$$L_n = AA', L_{2n} = AB$$
$$L_{2n}^2 = (\frac{L_n}{2})^2 + (1 - OC)^2$$
$$= \frac{1}{4}L_n^2 + \left(1 - \sqrt{1 - \frac{1}{4}L_n^2}\right)^2$$
$$= 2 - 2\sqrt{1 - \frac{1}{4}L_n^2}$$
$$L_{2n} = \sqrt{2 - \sqrt{4 - L_n^2}}$$

假定圆的半径$r = 1$, 则有正6边形的边长 $L_6 = 1$, 利用迭代关系式可以快速求出周长
$$C_O = 2\pi = \lim_{n\to\infty}(n \times L_n)$$
$$\pi = \frac{1}{2} \times \lim_{n\to\infty}(n \times L_n)$$

上述算法收敛性很快！

在GeoGebra中实现起来也很方便。

1. 先建立迭代次数滑动条 $n = slider(1, 10, 1)$
2. 再建立迭代函数 $f(x) = sqrt(2 - sqrt(4 - xx))$
3. 然后用GGB的迭代命令 $value = Iteration(f, 1, n) \to L_{2n} = \sqrt{2 - \sqrt{4 - L_n \times L_n}}, L_6 = 1$，初始值取正6边形时的值1.
4. 正2n边形的半周长为 $\pi = value \times 6 \times 2^{n-1}$

这一算法的收敛也极快，正多边形的边数以指数级增长。边数 $= 6 \times 2^{n-1}$