PROJECTILE MOTION *Minimum velocity*

Suppose we want to throw a baseball from deep center field to second base. If a runner is advancing, we want to throw on a trajectory that minimizes the TOF. Intuitively, this would be the shortest flight path, or arc length of the trajectory parabola. We know that there are two angles to reach a given "target", but one will have a longer arc length than the other. Certainly, minimizing the arc length would be one way to minimize the TOF, but it's far easier to just consider the general TOF for the "target" problem, namely

$$T = \frac{X}{v_0 \cos(\theta)}$$

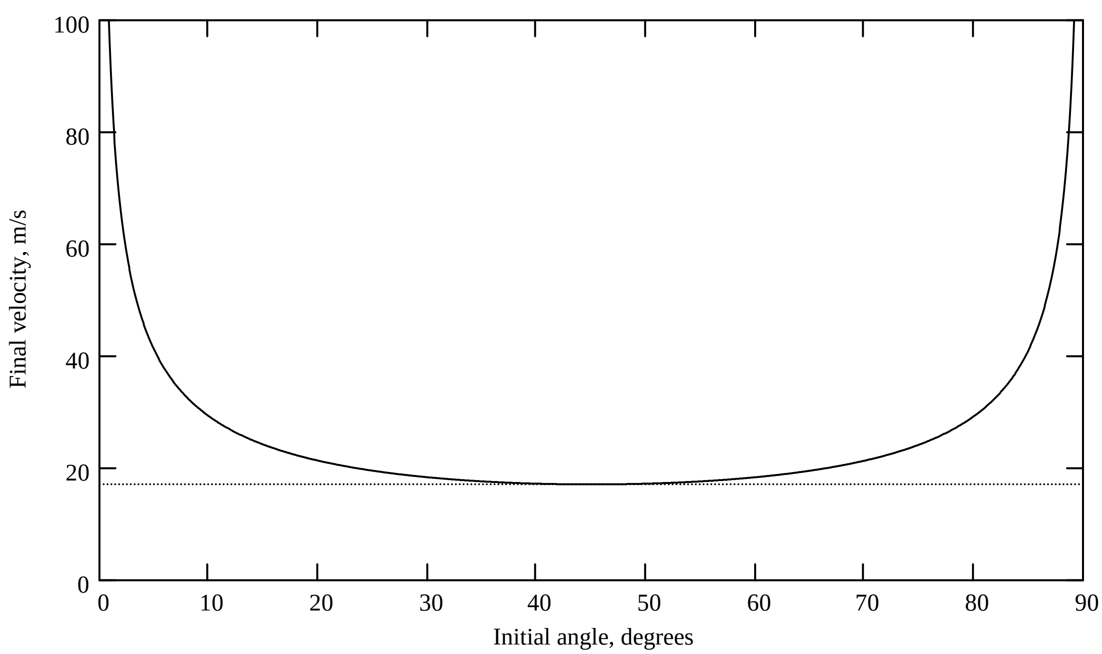
This says that, to minimize T, we want to maximize cos(θ), which means to minimize the initial angle θ. Then we would also want to maximize the initial velocity. So, as common sense says, we would throw a "frozen rope" to second base.

Now suppose that you are playing catch and there is a young child at second base and you want to throw to her from center field in such a manner that the final velocity of the ball as it reaches the base is minimized. Certainly the "frozen rope" is not a good choice, but how about a high "lob" (moonball)? We know that the final velocity is

$$v_f = \sqrt{v_0^2 + 2\ g\ y_0}$$

and for all practical purposes, the initial height is zero, so the final and initial velocities are equal. Thus the problem is to find the minimum initial velocity that can attain a given range (distance from you to second base). The range for this case is

$$R = \frac{v_0^2}{g} \sin(2\ \theta) \qquad \text{so that} \qquad v_0(\theta) := \sqrt{\frac{g\ R}{\sin(2\ \theta)}}$$



From the plot we see that the initial (and, thus, final) velocity is symmetric, so that we have two angles that will produce the same velocity. These are of course the Galileo angles, and they are complementary. There is a broad minimum in this function, so that across a wide range of angles, the velocity does not change much. We can find the minimum of this function using calculus:

$$\frac{dv_0}{d\theta} = \frac{-g\ R\ \cos(2\ \theta)}{\sin(2\ \theta)^{\frac{3}{2}}}$$

For a critical point this derivative will be zero. Since the only thing that can be zero here is cos(2θ), we will have the extreme point at 45 degrees, where the velocity will be

$$v_{min} = \sqrt{g\ R}$$

As indicated in the plot, if we throw the ball at a shallow angle, the velocity increases. But it also increases if we "lob" it at a high angle. **The final velocity does not depend on the initial angle**. This velocity is necessary to get the ball to cover the distance in the x direction just as it hits the ground. So, to minimize the velocity for a *given* distance, we throw the ball at a 45 degree angle, and if this velocity is still too high for the child to safely catch it, we must move closer to second base, reducing the range R.