Using GeoGebra to Improve Trigonometry Instruction

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Abstract:

A course in trigonometry is an important step in the sequence of higher mathematics education. Research indicates that trigonometry is a subject where students face significant difficulty in gaining understanding, especially in more traditional, lecture-based learning environment. However, applying the mathematical software GeoGebra to provide student-centered, discovery-based trigonometry lessons has been shown to be effective in addressing these difficulties and achieving better learning outcomes. In the following paper, a collection of GeoGebra applets- containing selections from other GeoGebra authors as well as original applets- is presented. The applets chosen and created exhibit features and design that align with instructional approaches found to be effective in the research. Student activities that supplement the applets in the collection are included in the appendix.

Introduction:

A course in trigonometry is an important step in the sequence of higher mathematics education. A thorough understanding of trigonometric functions can serve as a foundation for modeling real world phenomena studied in fields including engineering, physics, and biology. Without a firm grasp on trigonometric concepts, a student would not fully realize the depth and value of the results yielded when calculus is applied to the concepts. However, research indicates that trigonometry is a subject where students face significant difficulty in gaining understanding, especially in more traditional, lecture-based learning environments. (Yilmaz; 2012) A general approach to combat this pattern is the use of mathematical software to develop student-centered, investigation-based lessons to facilitate deeper learning of trigonometry.

In particular, several studies found that using the software package GeoGebra as a primary means of instruction in a trigonometry course leads to a greater impact on learning as compared to a control group. (Demir, 2012; Yilmaz, 2012; Ross, 2011) GeoGebra is an open source dynamic geometry software and computer algebra system in which students and teachers can use a user-friendly interface to create dynamic mathematical objects and illustrations. GeoGebra applets can be created with little programming background, allow for easy interaction and manipulation of geometric objects, algebraic expressions, functions, variables, and parameters. The applets can be published, downloaded and modified by users in the GeoGebra network.

Applying GeoGebra to trigonometry instruction is particularly valuable, as studying the important trigonometric objects and processes in a geometric, algebraic, and graphical context is vital to a deep understanding. (Demir, 2012) In this project, a collection of GeoGebra applets chosen to support instruction in a college trigonometry course is presented. Several applets are original designs, while others are creations published in the GeoGebra network by other authors, or modified versions of these creations. Each applet is presented and discussed in terms of its potential use as a digital manipulative in a student-centered, exploration-based lesson to facilitate a more integrated and thorough understanding of trigonometry. Several applets include original student activities that are included in the Appendix.

The resources created and chosen cover the following topics: (1) Right triangle definitions of trigonometric functions, (2) unit circle definitions of trigonometric functions, (3) geometric discovery of trigonometric identities, (4) connection between the unit circle and graphs of transformations of sine, cosine, and tangent and (5) geometric proofs angle sum and multiple angle identities, and (6) dynamic illustrations of two right triangle word problems.

Note that each of the applets are designed so that students are not required to program or create objects, but rather to manipulate existing objects, keeping the focus only on the mathematical concepts at hand. The user works within the designed environment where they can investigate and make discoveries within a certain mathematical context. The applets in the collection were created or selected based on principles and approaches to teaching and learning supported by the research.

Research:

In considering how GeoGebra can impact trigonometry instruction specifically, it is helpful to first consider its role as a learning tool in a broader sense. GeoGebra can be classified as a virtual manipulative, as it can be used to generate replicas of real-world manipulatives with which students can interact and explore. Virtual manipulatives are effective in engaging and motivating students, as well as supporting students to make conjectures and discoveries. GeoGebra's free and simple interface, high resolution, colorful display, ability to provide immediate feedback, and ability to visualize abstract and complex mathematical objects makes it an impactful learning tool. (Roblyer, 2018).

The applets in the collection make use of these features of GeoGebra to increase motivation and provide a richer learning environment. First, the applets allow the student to easily engage with new content and to be less reliant on an instructor. By following prompts to interact with free objects and sliders and observing patterns and relationships, the student assumes a more active role. According to the Constructivist perspective of learning, new knowledge is constructed actively by an individual and not transmitted to the individual by another source. A GeoGebra applet facilitates the application of this principle, serving as a laboratory in which students can build upon or adjust the schema targeted by the applet through assimilation and accommodation. (Demir, 2012)

A specific and important aspect of the process of knowledge construction is discovery. According to Trung (2014), the concept of discovery learning is defined as "learning with students building their own knowledge by experimenting and drawing conclusions of rules/concepts from their experimental results." In addition to supporting knowledge construction, discovery learning is a satisfying experience that can intrinsically motivate students to explore and investigate (Trung, 2014). In a study by Juandi (2018), GeoGebra was implemented in a discovery learning model, and it was found that the software impacted the students' ability to use visual thinking to make discoveries.

Like other educational software tools, a primary strength of GeoGebra is its ability to support cognitive processes. According to Demir (2012), GeoGebra can be considered as a cognitive tool, in part because of its potential to "share the cognitive load by providing support for lower level cognitive skills so that resources are left over for higher order thinking skills; allow the learners to engage in cognitive activities that would be out of their reach otherwise."

For example, an applet was used to prompt students to move a dynamic point around a polygon and observe its vertical position. In this context, GeoGebra removed the cognitive constraint of visualizing and tracking the position of a continuously moving point by hand, but kept the student in control of the investigation, performing only a part of the process necessary to explore the concept. Rather than only demonstrating a concept, the applet allowed the student to "observe, realize, and investigate mathematical relationships," which is valuable to learning.

Having considered the general value of GeoGebra as a learning tool demonstrated in the research, its ability to address specific challenges in trigonometry instruction will now be discussed. A typical framework for a trigonometry course consists of studying trigonometric functions in 3 ways: Right triangle trigonometry, Unit circle trigonometry, and trigonometric functions of real numbers. A common problem for college level trigonometry students is a disconnected knowledge of the three domains, and a lack of an integrated understanding between them (Demir, 2012). According to Orhun (2001), understanding the relationship between the unit circle and trigonometric graphs is important for a robust understanding of functions, but students in a study had significant difficulty making this connection. Furthermore, trigonometric functions cannot be calculated arithmetically and are instead formulated as a geometric object or process, which leads to difficulty in evaluating and graphing the functions. (Demir, 2012).

An important result of the study by Demir is that the use of GeoGebra helped students to gain stronger conceptual understanding of right triangle trigonometry, unit circle trigonometry, and real valued trigonometric functions and their graphs. Additionally, using GeoGebra resulted in students having a deeper understanding of the connections in each of the three domains. However, the study included only a few examples of applets used for instruction. In the following section, the collection of applets is presented to demonstrate how GeoGebra can be used to enhance instruction in a trigonometry course as described above.

Instructional Materials:

Right Triangle Definitions of Trigonometric Functions

The first context in which students encounter trigonometric functions is in terms of ratios of sides of a right triangle relative to a chosen angle. The traditional approach to introducing the definition of the sine of an angle, for example, is as the quotient of the opposite side and the hypotenuse. This approach focuses on the arithmetic computation of sine, as students may calculate ratios for triangles with given side lengths. However, it does not promote an understanding of the geometric nature of the concept of sine. Understanding trig functions as geometric processes rather than arithmetic formulas is important for students to build a deeper and more coherent understanding of the functions. (Demir, 2012).

As an alternative approach, students can use the GeoGebra applet "Conceptual Illustrations: Trigonometric Ratios within Right Triangles" (Brzezinski, 2018a) to construct a more intuitive and geometric notion of sine, cosine and tangent as ratios.

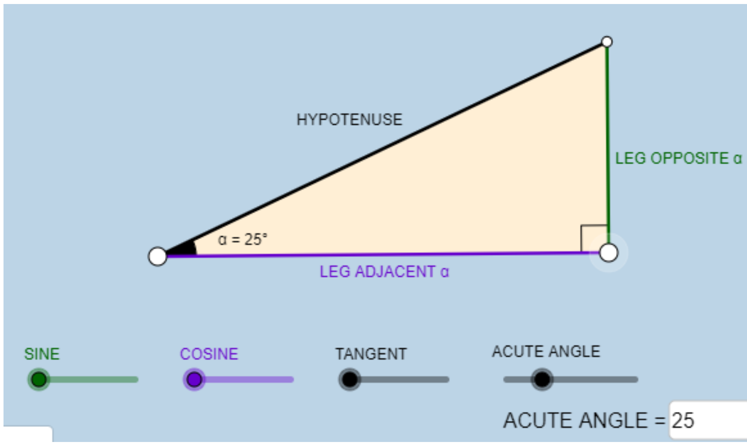

Figure 1: Trig Ratios within Right Triangles

The applet interface is shown in Figure 1. A right triangle with a dynamic acute angle alpha and movable vertices (large white points) is displayed. The user is free to adjust the orientation and angle of the triangle to consider a continuum of examples quickly and easily, which is conducive to discovery learning. Furthermore the triangles' side lengths are not given, reinforcing the geometric conception of sine, cosine, and tangent. Rather than computing the ratios of side lengths, students can use a slider for each of the functions to animate the triangle and visually explore the ratio between the side lengths. A video tutorial is included on the GeoGebra webpage to guide students' use of the applet.

Figure 2 shows the result of moving the "Sine" slider from Figure 1. The slider rotates the opposite side of the triangle about its top vertex so that it is coincident with the hypotenuse. This animation allows the student to easily compare the length of the opposite and hypotenuse, and the text prompts the user to observe that the opposite side covers about 42% of the hypotenuse

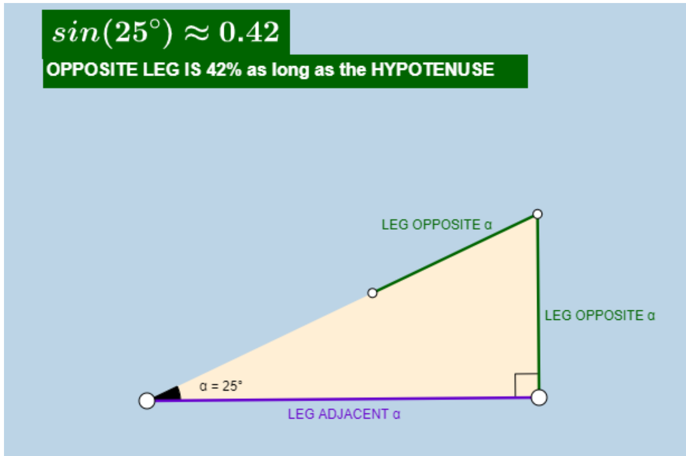

Figure 2: Conceptual Illustration of Sine

This guides the user to construct an intuitive definition of sine and helps the student to estimate the value of sine without knowing the side lengths or performing computation. At any point, the user may adjust alpha and explore how this percentage changes as alpha varies. For example, student can explore special angle triangles by setting alpha to 30, 45, or 60 degrees and observing sine, cosine, and tangent for each angle. Furthermore, students can use this feature to make a conjecture on the values of sin(0) and $sin\left(\frac{\pi}{2}\right)$, despite the right triangle disappearing for these values. This applet highlights the ability of GeoGebra as a cognitive tool and a facilitator of exploration and discovery.

Unit Circle Definition of Trigonometric Functions

Understanding trigonometric functions in the context of the unit circle is important in connecting the three domains of trigonometry (Demir, 2012). Once students can establish that

$(sin(\theta), cos(\theta))$

are the coordinates of a point on the unit circle rotated by θ counterclockwise from the x axis, they can use prior geometric knowledge to make connections and discoveries about the relationships between trigonometric functions. The applet "New Trig IDs from Similar Right Triangles v2" by Brzezinski (2019) can be used to facilitate a geometric discovery of the 6 trigonometric functions as the lengths of segments within the unit circle construction. For this project, the applet was modified to include labeling of key points in the construction and animations for similar triangles in order to provide more support to students in making discoveries.

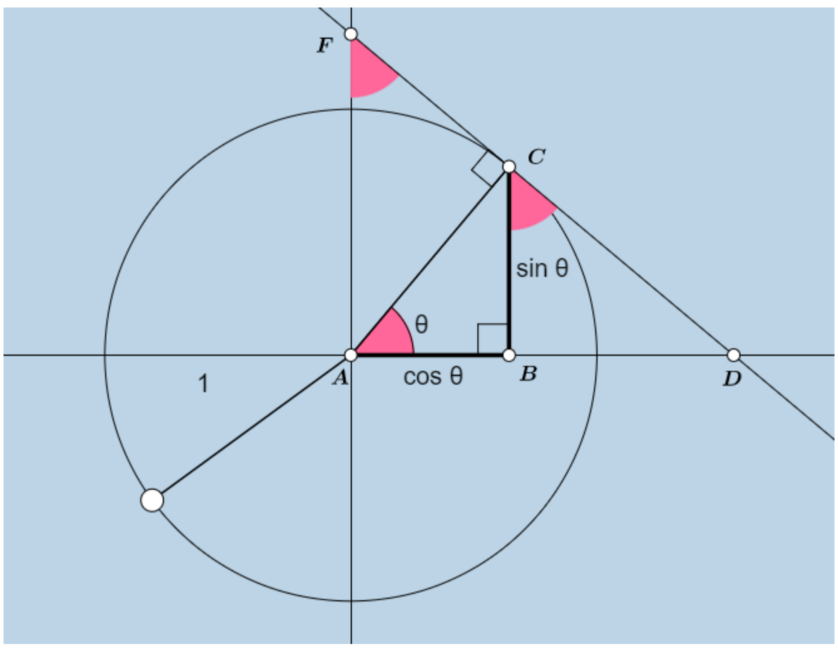

Figure 3: Geometric Construction of Trig Functions from Unit Circle

Figure 3 displays the applet and the initial construction. Point C is rotated by θ radians about the x axis, and a radius AC of the unit circle is drawn, and point B is the projection of C onto the x axis. The angle θ is dynamic in the applet. A tangent line to the circle through C is drawn, and the points D and F are the intersections of the tangent line and the x and y axis, respectively.

First students can apply right triangle trigonometry to ΔABC to establish AB = cos(θ) and BC = sin(θ). The pink angles in the applet indicate which angles are congruent to θ and thus allow the student to identify similar triangles. Students can use a slider to adjust θ and observe that the triangle similarity is not dependent on θ. Using these facts and investigating the proportions of similar triangles, students can discover sec(θ), tan(θ), csc(θ), and cot(θ) as lengths of segments in the construction. The check boxes and "hint" slider lets the student view animations to compare similar right triangles and prompts them to write proportions to discover the segment lengths, as shown in Figure 4. The student activity "Trig Functions and the Unit Circle" found in the Appendix guides students through the exploration.

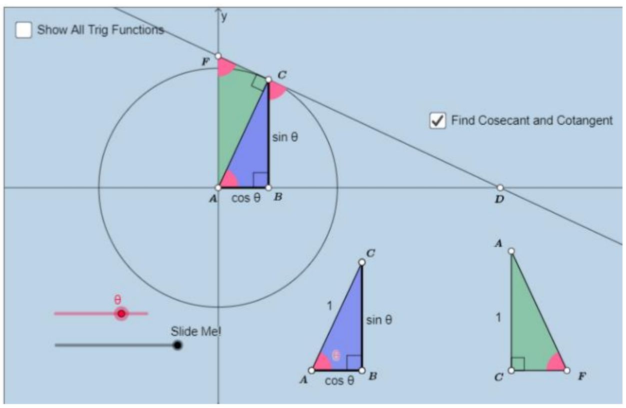

Figure 4: Discovering Trig Functions

The applet also has a checkbox to reveal the lengths of each segment for students to verify their discovery. This activity allows students to make connections between the notation and trigonometric functions. For example, students can observe that tangent and cotangent are represented as segments of the tangent line, whereas secant and cosecant are segments of secant lines. Also, students can again vary the angle θ and investigate the lengths of each segment as θ varies between 0 and $\frac{\pi}{2}$ radians, which later allows for insight into graphing the trigonometric functions. For example, rotating θ to $\frac{\pi}{2}$, the student may notice that the tangent segment cannot be drawn, giving a geometric interpretation of $\tan\left(\frac{\pi}{2}\right)$ being undefined.

Geometric Discovery of Trigonometric Identities

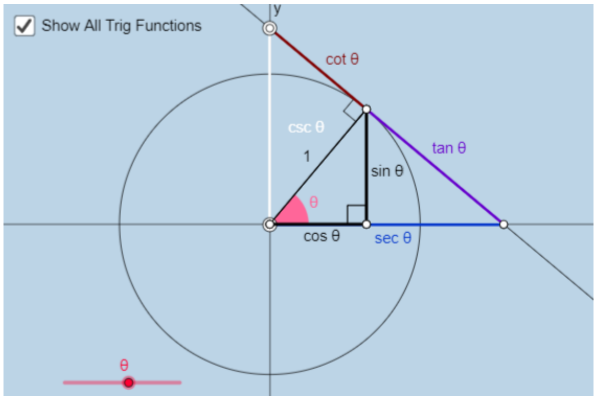

Figure 5: Discovering Trig Identities

Finally, the applet is also useful in providing a visualization of trigonometric identities, which is an alternative to studying identities in a purely symbolic way. By clicking the "Show All Trigonometric Functions" checkbox as shown in Figure 5, each trigonometric function is represented as a segment in the construction. By interacting with the applet, the student can observe that the geometry – in particular, the right triangles – is preserved as θ changes. By applying the Pythagorean theorem to the preserved right triangles, students can discover the Pythagorean identities from the applet and visually verify that they hold no matter the choice of θ. Furthermore, other geometric formulas that are familiar to the students can be applied to the construction to allow students to write down new, more elaborate identities which can then be verified symbolically. The student activity "Discovering Trig IDs" included in the Appendix prompts students through the investigations described above.

Connecting the Unit Circle to Graphs of Sine, Cosine, and Tangent

Graphs of trigonometric functions in the plane is the primary tool used for extending trigonometric functions based on the unit circle to trigonometric functions of real numbers. (Weber, 2005). Trigonometric functions of real numbers and transformations of these functions are prevalent in many mathematical and scientific disciplines; thus it is important for students develop understanding in this part of the framework. However, this tends to be the area where students tend to have the most difficulty building knowledge (Yilmaz, 2012). However, designing instruction to make connections between unit circle geometry and trigonometric graphs was found to positively impact learning in this real-valued function domain and facilitate a more integrated understanding between the three domains Demir, 2012).

Because of its capability to be used as a graphing calculator and a dynamic geometry software simultaneously, GeoGebra is an effective way to apply this approach to instruction. The GeoGebra applet "Unit Circle to Sine and Cosine Functions" by Brzezinski (2017) demonstrates how GeoGebra can help students make a geometric-to-graphical connection for the sine and cosine parent functions.

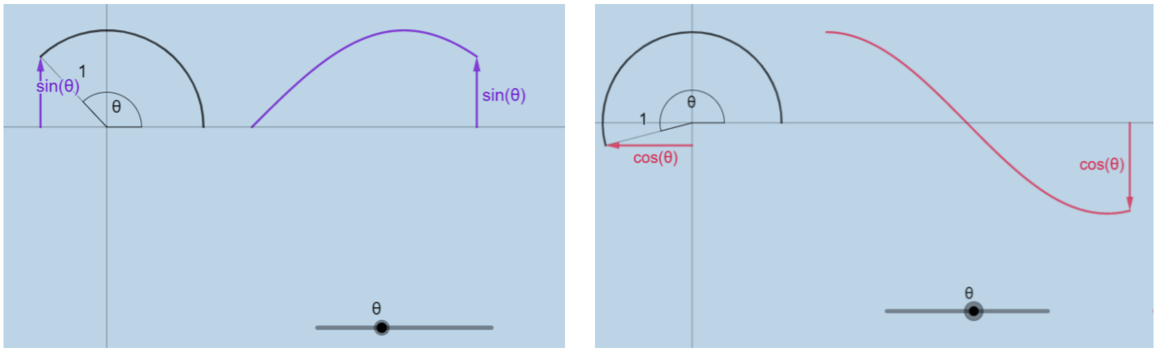

Figure 6: Unit Circle to Function Relationship for Sine and Cosine

Figure 6 displays both options for the sine and cosine functions. The applet features an illustration of a unit circle with sine (cosine) as a directed line segment on the left, and on the right, the corresponding curve. This applet can serve as an aid in conceptualizing the input and out of the sine and cosine functions. The student can see that the sine (cosine) function is tracking the vertical (horizontal) position of a point as it is rotated along the unit circle by θ. The applet features an animation to clearly demonstrate that the y- coordinate on the function is determined by the directed length of the corresponding segment. However, the applet does not show a clear connection between the angle and the x-coordinate of a point on the wave.

Furthermore, the applet is restricted to an investigation of only the parent functions for sine and cosine. However, GeoGebra was also found to be useful impacting student learning in graphing transformations of trigonometric functions (Ross, 2011). In the study, students used GeoGebra to explore connections between the algebraic representation of the function and its graph, ignoring the geometric connection. Thus, this project includes GeoGebra applets that allows for exploration of transformations of the sine, cosine, tangent function that is based on a geometric connection to the unit circle. For simplicity, the following will focus on the transformations of sine applet.

A typical study of transformations of trig functions is focused on the equation of the form $\boldsymbol{y = r sin(b(\theta - c)}$**,** and students a investigate how each of the parameters a,b, c, and d impact the shape of the function relative to the parent graph. In the geometric approach to transformations of this form, the parameter d was ignored due the simplicity of its effect (vertical shift of the curve by d units) relative to the other parameters. However, investigating the geometric effect of a, b, and c allows students to gain deeper insight that into the function transformations and appeals more to prior knowledge as compared to a strictly algebraic investigation.

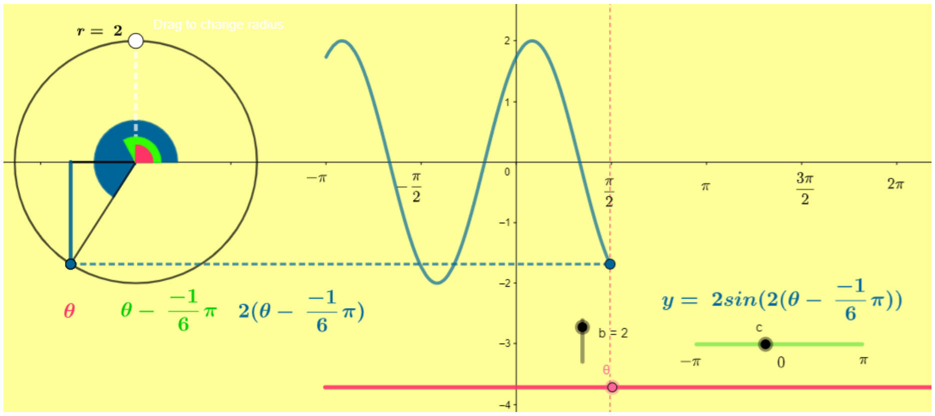

Figure 7: Geometric Transformations of Sine

The app “Transformations of Sine” ,shown in Figure 7, has a design similar to “Unit Circle to Sine and Cosine Functions”. The circle that generates the curve is shown simultaneously with the graph of the function on separate axes, and the input $\boldsymbol{\theta}$ is determined by a slider. However, notice that the horizontal position of the slider corresponds to the measure of the pink angle to establish the connection between the angle measure being the input, or x-coordinate, on the function.

To account for transformations, this applet contains sliders for b and c. Also note that the radius of the circle representing the parameter r is modifiable by dragging the gray point. The parameters b and c transform the input angle $\boldsymbol{\theta}$**.** Students can use their prior knowledge about angle measure observe that the green angle is $\boldsymbol{\theta - c}$ and the blue angle is b($\boldsymbol{\theta - c}$**).** For example, using order of operations as expressed in the function algebraically, the student can track the transformation of theta: first it is moved counterclockwise by c = $\frac{\pi}{6}$ radians to produce the green angle, and then the green angle is multiplied by b = 2, or doubled, to produce the blue angle. Finally, the applet demonstrates the evaluation of y = 2sin(b($\boldsymbol{\theta}$ – $\boldsymbol{c}$)), by tracking the y coordinate of the point associated with the transformed angle along a circle of radius 2, and plotting it on the function as the y coordinate corresponding with the measure of the pink angle $\boldsymbol{\theta}$.

This applet shows GeoGebra's value as a cognitive tool; The students can simply observe the angle transformations visually, allowing for more focus on the resulting transformation of the function without losing sight of the effect of the geometric effect of the parameters b and c. For example, the green arrow shown in Figure 8 demonstrates the resultant transformation caused by subtracting c = $\frac{\pi}{2}$ from $\boldsymbol{\theta}$**.** Figure 8 also demonstrates how the applet facilitates discovery of the connection between a negative b value (b = -1), clockwise rotation of the blue point, and a reflection of the parent graph of sine. Finally, by shrinking the or stretching to radius to adjust the r parameter, students

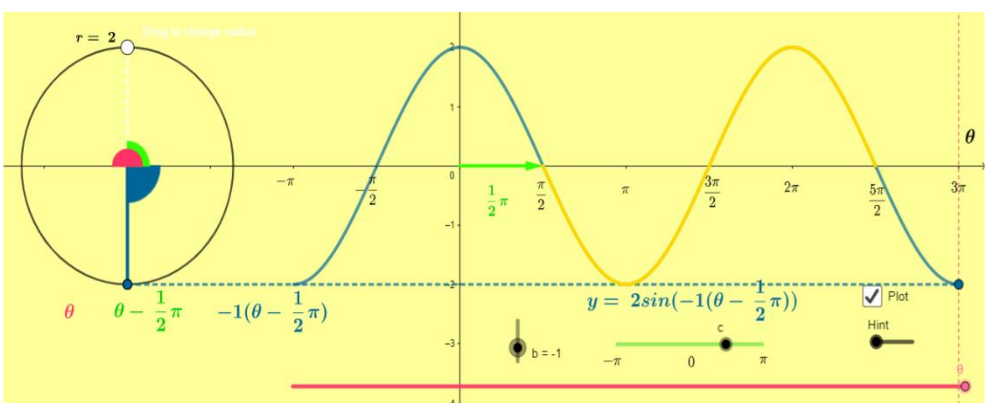

Figure 8: Result of Transformations of Sine Graph

can intuitively relate the radius of the circle and the amplitude of the wave. The activity "Transformations of the Sine Function" in the Appendix can be used to guide student exploration of the effects of each parameter on transformations. Figure 9 displays the "Transformations of Tangent" applet". For the tangent function, the geometric representation is given by the tangent segment to the circle as demonstrated in applet 2.

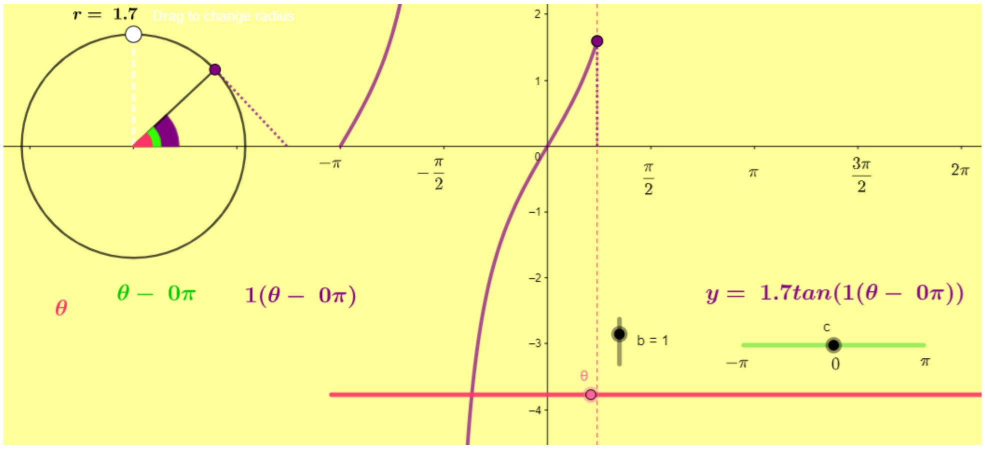

Figure 9: Transformations of Tangent

Geometric Proofs of Angle Sum and Multiple Angle Identities:

In the study of trigonometric functions of real numbers, a main area of focus is trigonometric identities and their applications to solving equations or manipulating trigonometric expressions. The identities studied typically include angle sum and difference formulas and double angle formulas for sine, cosine, and tangent. Instruction in this domain tends to be focused on developing students' fluency in working symbolically with these identities in preparation for calculus. Designing instruction focused mainly on symbolic manipulation of trigonometric functions may suggest a disconnect from the geometric domain and contribute to students fragmented view of the subject.

GeoGebra can be used to help students discover underlying geometric relationships in studying trigonometric identities. Specifically, dynamic applets can be used to guide students to construct novel geometric arguments of common identities that are based in prior knowledge. Two applets will be presented in which students construct a proof of the angle sum formula for sine, and the double angle identity for sine, respectively.

The applet "Sine and Cosine of a Sum: Discovery" by Brzezinski (2018b) is an activity in which students use right triangle trigonometry and geometric reasoning to provide a visual proof of the angle sum formulas for sine and cosine. Figure 10 displays the set-up of the proof. By using right triangle definitions for sine and cosine of α, β and $\alpha + \beta$, students can match segments forming the 4 right triangles in the construction with the corresponding expression on the right. To finish the proof, they are prompted to equate the expressions for each pair of parallel sides of the rectangle to discover the identities. The applet reinforces the meaning of identity, as the user can drag the purple point and discover that the expressions for each of the segments do not change as the angles change. A video demonstration of the proof is included on the GeoGebra webpage to support students' use of the applet.

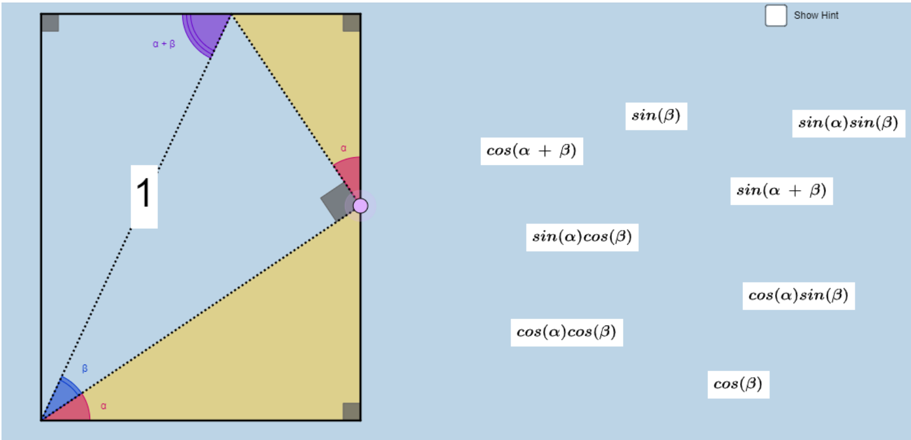

Figure 10: Sine and Cosine Sum Discovery

The second applet, “Geometric Proof for the Double Angle Sine Identity” makes use of the unit circle definition of sine and cosine to provide a geometric proof of the identity. The corresponding activity in the Appendix helps the student to construct an argument based on expressing the area of a triangle in two different ways.

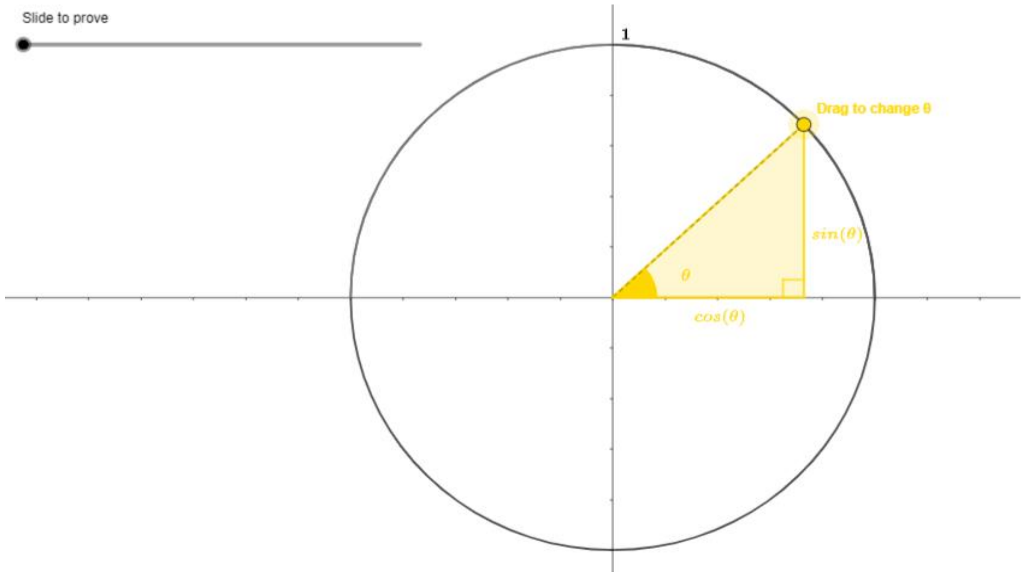

Figure 11: Geometric Proof of Double Angle Sine Identity (Setup)

First, the student is instructed to choose an arbitrary angle $\boldsymbol{\theta}$ by dragging the yellow point, indicating that the following result is independent of the value of $\boldsymbol{\theta}$. Then, they are instructed to move the slider to illustrate the proof, stopping several times to observe the figure and write down important expressions. GeoGebra is used to animate a rotation and reflection of the yellow triangle shown, which facilitates the proof. The proof builds on students’ knowledge of triangular area and the fact that area is preserved by reflection and rotation. Students can write down expressions for the yellow and blue triangles shown in Figure 12 and observe that the triangles have equal area. Equating the area of the regions, students discover the identity. The student activity “Proof of the Double Angle Sine Identity” included in the Appendix can be used to guide proof using the applet.

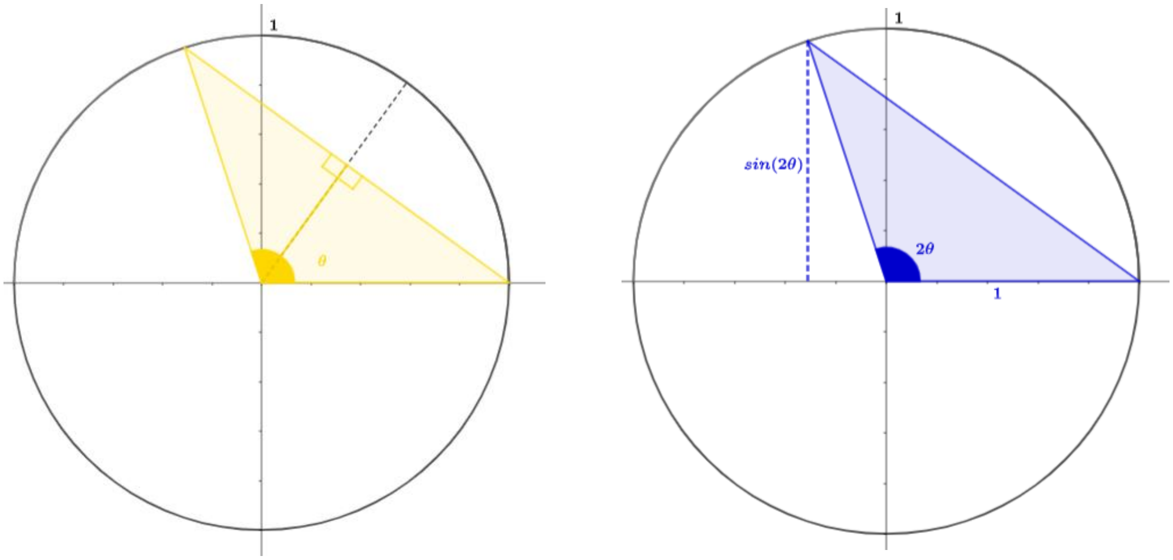

Figure 12: Geometric Proof of Double Angle Sine Identity by Equal Areas

This applet again demonstrates GeoGebra in the capacity of a cognitive tool. While reflections and rotations in the plane are difficult to perform by hand, they are easily animated within GeoGebra, allowing students to explore the resultant figures and think mathematically to write down expressions for area. Additionally, the dynamic angle $\boldsymbol{\theta}$ allows the student to explore this visual proof for any angle between 0 and π. For different choices of $\boldsymbol{\theta}$**,** the proof uses either an obtuse or an acute triangle. This demonstrates different possible cases within the proof and allows the opportunity to challenge students to discover when and how each case occurs. Proof by cases is an important mathematical idea that is difficult to illustrate without the software in this context.

Dynamic Illustrations of Right Triangle Word Problems

During instruction on right triangle trigonometry, a common objective is for students to apply definitions of the 6 trigonometric functions to solve real world problems. In this context, GeoGebra can be used to implement a dynamic illustration of a word problem. GeoGebra features such as dynamic construction, labeling, and coloring of objects, and the ability to show and hide objects make it a valuable scaffolding tool for students. For example, in a trigonometry problem, these features can be used to highlight similar triangles' congruent angles, and allow students to explore relationships between side lengths and angles as their measures are changed. Two GeoGebra illustrations of right triangle trigonometry word problems from *Algebra and Trigonometry* by Larson (2013) are presented and discussed below. The two exercises selected are problems with which students most commonly needed support and assistance during tutoring sessions.

Consider the problem shown in Figure 13. (Larson, 2013): To obtain a solution for the distance between the ships, the student must solve for the distance to the shore for both ships using right triangle trigonometry, and then find the difference of these distances between each ship and the lighthouse.

24. DISTANCE An observer in a lighthouse 350 feet above sea level observes two ships directly offshore. The angles of depression to the ships are 4° and 6.5° (see figure). How far apart are the ships?

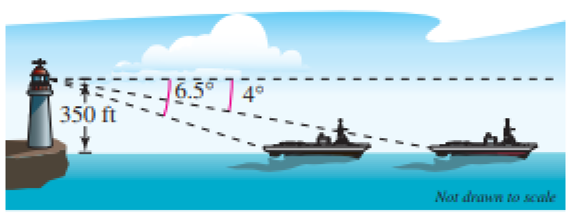

Figure 13: Exercise 6.7.24, *Algebra and Trigonometry* (Larson, 2013)

Using show/hide checkboxes in GeoGebra, each part of the problem can be illustrated separately to clarify the problem, and then integrated to reach the solution.

In Figure 14, the “Distance between Ships” view is shown. The student can adjust the height of the light house and both angles of depression to the ships to gain insight on the variables affecting the distance between the ships. Also, it allows the student to explore the relationship between **a (**the distance from ship 1 and the shore)**, b** the distance (between ship 2 and the shore), and the distance **b – a** (distance between the ships).

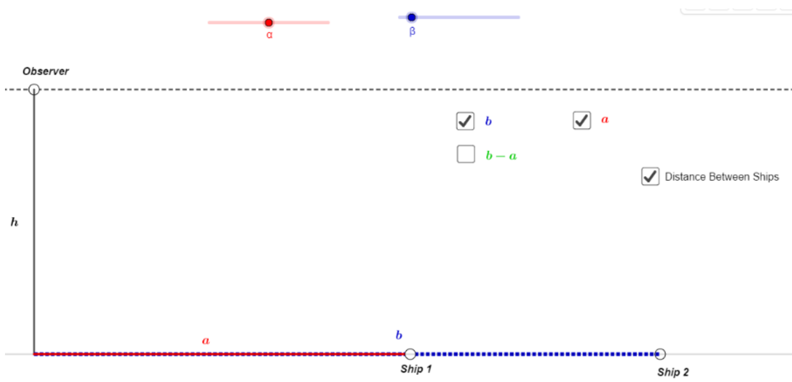

Figure 14: GeoGebra Illustration of Exercise 6.7.24

Once the relationship is established, the student can switch views to explore the right triangles that include **a** and **b**, as shown in Figure 15. GeoGebra is useful in illustrating the congruent alternate interior angles in each triangle. The student can then consider each triangle to write an expression for **a** and **b** using tangent of the red and blue angles. The applet provides support by outlining the problem-solving process and allowing students to easily toggle between views to clearly visualize the problem.

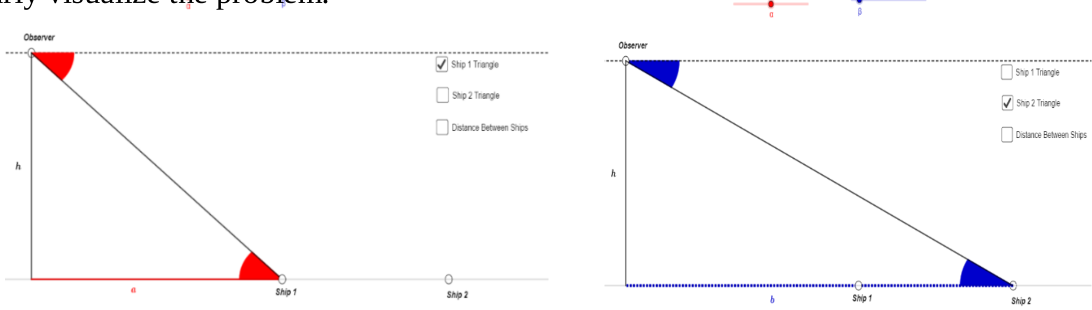

Figure 15: Alternate views of Exercise 6.7.24

The second problem illustrated in GeoGebra is Exercise 6.7.30 (Larson, 2013): The GeoGebra illustration of this problem, shown in Figure 16, allows the student to explore the geometry as the satellite moves about its orbital path by dragging the "Satellite" point. Rather than viewing a static illustration of the problem, the student can interact with a more realistic and

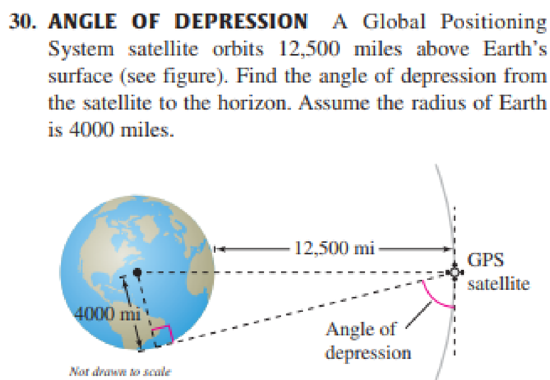
30. ANGLE OF DEPRESSION A Global Positioning System satellite orbits 12,500 miles above Earth's surface (see figure). Find the angle of depression from the satellite to the horizon. Assume the radius of Earth is 4000 miles.

Figure 16: Exercise 6.7.30

engaging model. Additionally, the student can observe that the right triangle, and thus the solution, is preserved as the position of the satellite changes, as shown in Figure 16. This can facilitate discussion and conjecture about why this is true and demonstrate to the student the effectiveness of trigonometry in solving dynamic real-world problems.

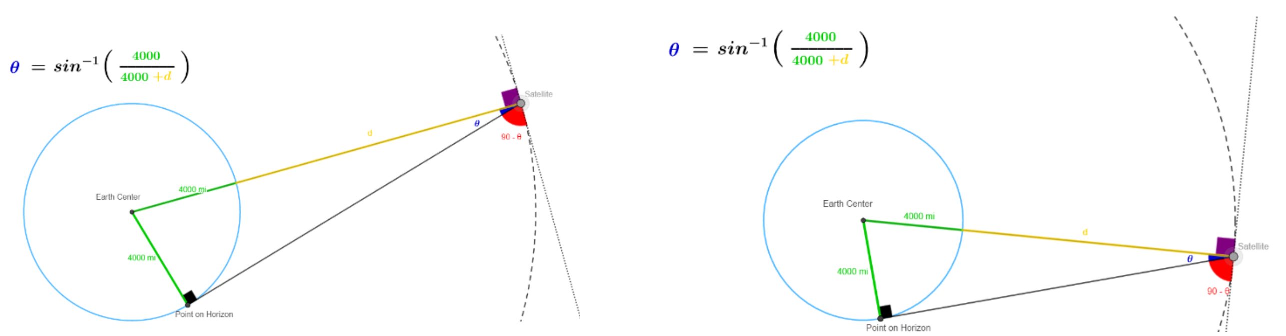

Figure 17: Dynamic Illustrations of Exercise 6.7.30

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Appendix

Activity: Trig Functions and the Unit Circle

Directions: Open the Applet "Discovering Trig Functions and Identities" to answer the following questions.

In the applet, the unit circle is shown. Point C is on the circle and FD is a tangent segment to point C. Recall that every tangent segment is perpendicular to the radius.

1. Notice that $< \boldsymbol{BAC} \;\; \boldsymbol{and} \;\; < \boldsymbol{ACB}$ are complementary. Also, $< \boldsymbol{ACB} \;\; \boldsymbol{and} \;\; < \boldsymbol{BCD}$ are complementary. What can we conclude about $< \boldsymbol{BAC} \;\; \boldsymbol{and} \;\; < \boldsymbol{BCD}$?

2. Observe that $< \boldsymbol{BAC} \;\; \boldsymbol{and} \;\; < \boldsymbol{FAC}$ are complementary. Also, $< \boldsymbol{FAC} \;\; \boldsymbol{and} \;\; < \boldsymbol{CFA}$ are complementary. What can we conclude about $< \boldsymbol{BAC} \;\; \boldsymbol{and} \;\; < \boldsymbol{CFA}$?

3. Based off the answers in 1 and 2, What can we conclude about the right triangles $\Delta\boldsymbol{ABC}$ **,** $\Delta\boldsymbol{ACD}$ and $\Delta\boldsymbol{FCA}$? (Hint: Consider AA similarity.)

Now, move the "Slide Me" slider to reveal the equal angles we just found. Notice lengths of the segments from the picture above are shown in the app. Also, notice that the triangles remain similar no matter the value of θ. ($< \boldsymbol{BAC} = \theta$).

4. Why is the length of segment AB = $cos\theta$? Why is the length of segment BC = $sin\theta$? (Hint: Remember point C is on the unit circle. Consider its coordinates.)

Now check the "Find Secant and Tangent" Check Box and answer the following questions.

5. Since $\Delta \boldsymbol{ABC}$ **and** $\Delta \boldsymbol{ACD}$ are similar, write a proportion of the side lengths of the triangles including the ratio $\frac{CD}{AC}$. (Use the hint slider to help write the equation). Solve for CD. Why is CD labeled $tan\theta$?

6. Next, use the same triangles to write a proportion of side lengths, this time including the ratio $\frac{AD}{AC}$. Solve for AD. Why is AD labeled $sec\theta$?

Now uncheck Find Secant and Tangent" and check the "Find Cosecant and Cotangent" check box and answer the following questions.

7. Since $\Delta \boldsymbol{ABC}$ **and** $\Delta \boldsymbol{FCA}$ are similar, write a proportion of the side lengths of the triangle. Include ratio $\frac{FC}{AC}$. Solve for FC. Why is FC labeled $cot\theta$?

8. Next, use the same triangles to write a proportion of side lengths, including the ratio $\frac{AF}{AC}$. Solve for AF. Why is AF labeled $csc\theta$?

Activity: Discovering Trig IDs

Directions: Open the Applet "Discovering Trig Functions and Identities" to answer the following questions.

In the Applet, check the "Show All Trig Functions" checkbox.

1. Apply the Pythagorean Theorem to each $\Delta \boldsymbol{ABC}$ **,** $\Delta \boldsymbol{ACD}$ and $\Delta \boldsymbol{FCA}$ separately. The equations you get are the Pythagorean Identities.

2. Apply the Pythagorean Identity to $\Delta \boldsymbol{FAD}$ to get a new identity. Prove this identity symbolically.

3. Notice that Area $\Delta \boldsymbol{FAD}$ = Area $\Delta \boldsymbol{ABC}$ **+**Area $\Delta \boldsymbol{CBD}$ +Area $\Delta \boldsymbol{FCA}$. Calculate the area of each triangle using its base and height and plug in to the equation above to derive another identity. Prove this identity symbolically.

4. Notice that $\Delta \boldsymbol{FAD}$ and $\Delta \boldsymbol{ACD}$ are similar. Write a proportion between side lengths to derive identity. Prove this identity symbolically. (AA Similarity)

5. Use the Pythagorean theorem, triangle similarity, or equal areas to find more identities. How many can you find?

Activity: Transformations of the Sine Function

Directions: Open the "Transformations of Sine" GeoGebra activity and use it complete the the following questions.

Intro: Remember the definition of sine, $\boldsymbol{sin\theta} = \frac{y}{r}$, where r is the radius of the circle, and y is the y coordinate of **the point on the circle AND the terminal side of $\boldsymbol{\theta}$.(The blue point).** Solving for y, we have $\boldsymbol{y} = \boldsymbol{rsin\theta}$, which represents the y coordinate of the blue point. For a given **r**, we can now graph the relationship by plotting points **$(\boldsymbol{\theta}, \boldsymbol{y})$.** On the horizontal axis is the **measure** of **$\boldsymbol{\theta}$,** and the vertical axis is **y** (vertical position of blue point). The collection of all ordered pairs together forms the graph of the function $\boldsymbol{y} = \boldsymbol{rsin\theta}$**.**

Directions: Make sure the checkbox for $\boldsymbol{y} = \boldsymbol{rsin}(\boldsymbol{b}(\boldsymbol{\theta} - \boldsymbol{c}))$ is checked. Set **b = 1**, **c = 0** (**b** and **c** have no effect for these values. We will explore these later). Uncheck **plot.**

1. Set **r = 1.** So our function is $\boldsymbol{y} = \boldsymbol{sin\theta}$**.** Use the pink slider to adjust $\boldsymbol{\theta}$ to each of the values in the table below**.** Record in the table the y coordinate of the blue point for each $\boldsymbol{\theta}$. Then plot the ordered pairs from the table on the axes below.

$\boldsymbol{\theta}$	$\boldsymbol{y} = \boldsymbol{sin\theta}$
0	
$\frac{\pi}{2}$	
π	
$\frac{3\pi}{2}$	
2π	

.

2. Now check **plot** to and slide θ to 3π to graph $\boldsymbol{y} = \boldsymbol{sin\theta}$ for $-\pi \leq \theta \leq 3\pi$. Sketch the wave on the axes above. What are the max and min y values of the function?

3. Now adjust to **r** = **2.** Sketch the function on the axes above. Then do the same for **r = .5**. What is the relationship between the minimum and maximum values of the function and **r?**

Directions: Again set **b = -1**, **c = 0.** Uncheck plot.

Note: In the circle, the blue angle is $-\boldsymbol{\theta}$, which is a clockwise rotation of the same amount as the counter clockwise rotation by the pink angle $\boldsymbol{\theta}$**.**

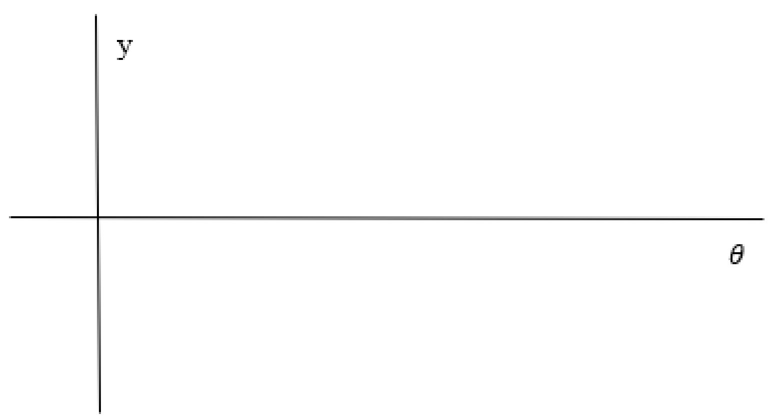
1. Set **r = 1**. Rotate $\boldsymbol{\theta}$ ***to*** $\boldsymbol{\frac{\pi}{4}}$ **.** Draw the circle with $-\boldsymbol{\theta}$**,** $\boldsymbol{\theta}$**,** and $\boldsymbol{sin}(-\boldsymbol{\theta})$ as it appears on the screen

2. Notice the y coordinate of the red point is $y = \boldsymbol{sin}(-\boldsymbol{\theta})$**.** Since sine is an odd function, we can also write

$$y = \boldsymbol{sin}(-\boldsymbol{\theta}) = \_\_\_\_\_\_\_\_$$

3. Now we can plot ordered pairs $(\boldsymbol{\theta}, \boldsymbol{y})$ for $y = -\boldsymbol{sin}(\boldsymbol{\theta})$**.** Use the pink slider to adjust $\boldsymbol{\theta}$ to each of the values in the table below**.** Record in the table the y coordinate of the blue point for each $\boldsymbol{\theta}$. Then plot the ordered pairs from the table on the axes below

$\boldsymbol{\theta}$	$\boldsymbol{y} = -\boldsymbol{sin\theta}$
0	
$\frac{\pi}{2}$	
π	
$\frac{3\pi}{2}$	
2π	



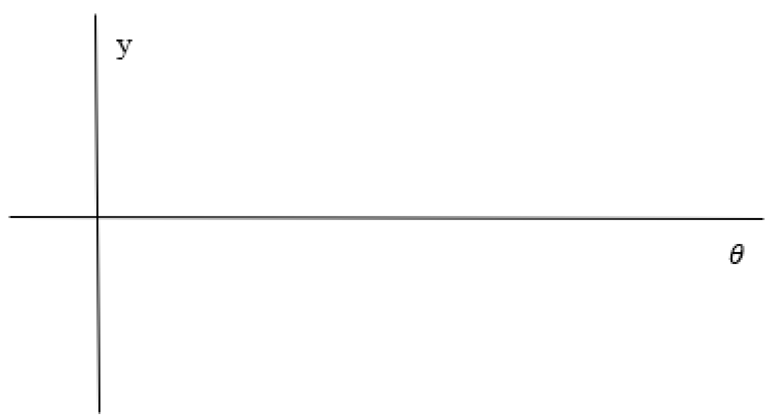
4. Now check **plot** and slide θ to 3π to graph $\boldsymbol{y} = -\boldsymbol{sin\theta}$ for $-\pi \leq \theta \leq 3\pi$. Plot the full wave on the axes above. Now adjust to the following values and plot the waves you see for each value: **r = 1 , r = 2, r = .5.** How do these functions compare with the sketches from page 1? What transformation is applied?

Directions: So far we have explored with **b = 1** and **b = -1**. In all sketches so far, notice that a full cycle (**highlighted in yellow)** has a horizontal width of 2π. The horizontal width of the of one cycle is called the **period** of the function. Observing the circle, the **period** is also the value of θ for which the blue point completes a full rotation around the circle.

1. Set b = 2. Slide $\boldsymbol{\theta}$ to different values. Notice the blue angle $2\boldsymbol{\theta}$ has a measure that is twice the pink angle. Set $\boldsymbol{\theta} = \frac{\boldsymbol{\pi}}{\mathbf{3}}$**.** Sketch the circle below showing $\boldsymbol{\theta}, \mathbf{2}\boldsymbol{\theta}, \ \boldsymbol{sin}(\mathbf{2}\boldsymbol{\theta})$

2. Notice the y coordinate of the blue point is now is $y = \boldsymbol{sin}(\mathbf{2}\boldsymbol{\theta})$**.** Use the pink slider to adjust $\boldsymbol{\theta}$ to each of the values in the table below**.** Record in the table y coordinate of the blue point for each $\boldsymbol{\theta}$. Then plot the ordered pairs from the table on the axes below

$\boldsymbol{\theta}$	$\boldsymbol{y} = \boldsymbol{sin2\theta}$
0	
$\frac{\pi}{4}$	
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	
π	



3. Now check **plot** to and slide θ to 3π to graph $\boldsymbol{y} = \boldsymbol{sin2\theta}$ for $-\pi \leq \theta \leq 3\pi$. Sketch the wave on the axes above.

4. Starting at $\theta = 0$, to what value do you slide θ so that the blue point makes one rotation around the circle?

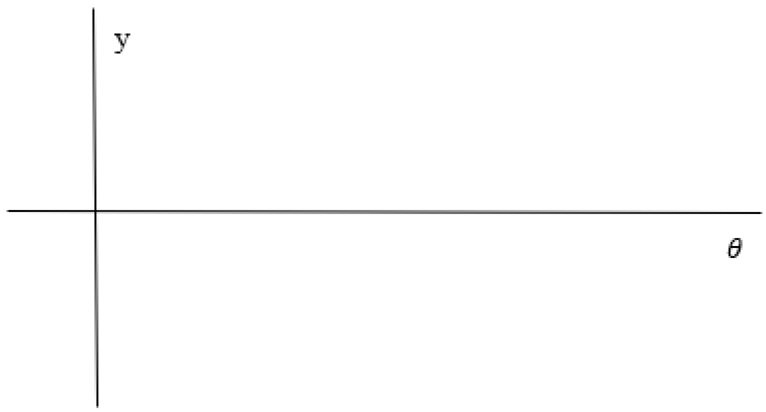
5. Remember to period is the horizontal width of a full cycle. What is the period of $y = \boldsymbol{sin}(\mathbf{2}\boldsymbol{\theta})$? **Set b = 3** and answer 3 and 4 again. What is the relationship between b and the period?

Directions: Set **b = 1 and r=1** and uncheck Plot.

Note: So far we have explored with **c = 0**. Slide $\boldsymbol{\theta}$ to 0 and now set $\boldsymbol{c} = \frac{\boldsymbol{\pi}}{\boldsymbol{6}}$. Now our equation is $\boldsymbol{y} = \boldsymbol{sin}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}}\right)$. Adjust $\boldsymbol{\theta}$ and notice that in the circle, the green angle is always $\frac{\boldsymbol{\pi}}{\boldsymbol{6}}$ radians behind $\boldsymbol{\theta}$. So the green angle has measure $\boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}}$.

1. Set $\boldsymbol{\theta} = \frac{\boldsymbol{\pi}}{\boldsymbol{3}}$ and sketch the circle showing $\boldsymbol{\theta}, \boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}}$, and $\boldsymbol{sin}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}}\right)$.
2. Notice the y coordinate of the blue point is now is $y = \boldsymbol{sin}(\boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}})$. Use the pink slider to adjust $\boldsymbol{\theta}$ to each of the values in the table below. Record in the table y coordinate of the blue point for each $\boldsymbol{\theta}$. Then plot the ordered pairs from the table on the axes below

$\boldsymbol{\theta}$	$\boldsymbol{y} = \boldsymbol{sin}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}}\right)$
0	
$\frac{\pi}{6}$	
$\frac{2\pi}{3}$	
$\frac{7\pi}{6}$	
$\frac{5\pi}{3}$	
$\frac{13\pi}{6}$	



3. Now check **plot** to and slide θ to 3π to graph $\boldsymbol{y} = \boldsymbol{sin}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}}\right)$ for $-\pi \leq \theta \leq 3\pi$. Sketch the wave on the axes above.

4. Notice the full cycle highlighted in yellow. Slide c back and forth between 0 and $\frac{\boldsymbol{\pi}}{\boldsymbol{6}}$. What transformation is applied to the highlighted cycle?

5. Now slide c to $-\frac{\boldsymbol{\pi}}{\boldsymbol{6}}$. What is the relationship between the pink and green angle? What transformation is applied to the highlighted cycle now?

6. What is the relationship between c and the graph of $\boldsymbol{y} = \boldsymbol{sin}\left(\boldsymbol{\theta} - \frac{\boldsymbol{\pi}}{\boldsymbol{6}}\right)$.

Activity: Proof of the Double Angle Sine Identity

The double angle sine identity is the following formula:

$$\boldsymbol{sin(2\theta) = 2sin(\theta)cos(\theta)}$$

Open GeoGebra App, "Geometric Proof of Double Angle Sine ID" to show a geometric proof of this identity. In the app, the unit circle is shown, with an angle θ in standard position. An identity equation is true no matter the value of the variable. Here the variable is θ. Drag the yellow point to pick any value for θ to begin the proof.

1. Notice the yellow triangle has a base of $\boldsymbol{cos(\theta)}$ and height of $\boldsymbol{sin(\theta)}$? How do we know this?

2. The area of any triangle is $\boldsymbol{A = \frac{1}{2}(base)(height)}$**.** What is the area of the yellow triangle?

3. Move the slider and notice the yellow triangle is rotating about the origin. Move the slider to position 1, then leave the slider in place. A rotation does not change the area of the triangle. Notice the triangle was rotated θ radians.

4. Now move the slider to position 2, then leave the slider in place. The original triangle was reflected about its base, and a copy of the original triangle is created. What is the total area of the two yellow triangles?

5. Now move the slider to position 3 and notice a single blue triangle appears. Notice this blue triangle overlaps perfectly with the two copies of the yellow triangle, so the blue area is the same as your answer to 4.

6. Move the slider to position 4 and notice the blue angle with its vertex at the origin appears. How do we know that its measure is 2θ?

7. Move the slider to position 5 and notice the height of the blue triangle is drawn by a dotted line. Why is the height of the blue triangle **sin(**$\boldsymbol{2\theta}$**)**?

8. Notice the base of the blue triangle is 1? How do we know this?

9. Calculate the area of the blue triangle using $\boldsymbol{A = \frac{1}{2}(base)(height)}$**.**

10. Recall the blue area was equal to the yellow area. So we can **equate** the expression of the yellow area (question 4) and the expression for the blue area (question 9). Write this equation below. Then multiply this equation through by 2 to complete the proof.