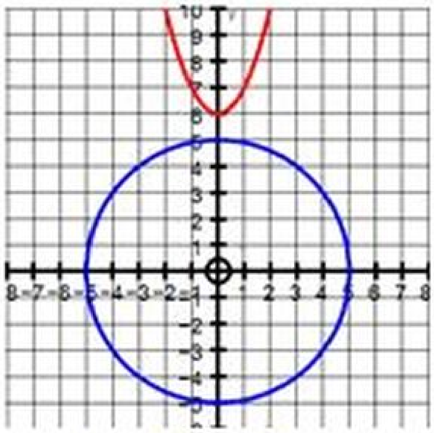
# Intersection Points of Circles with Parabolas.

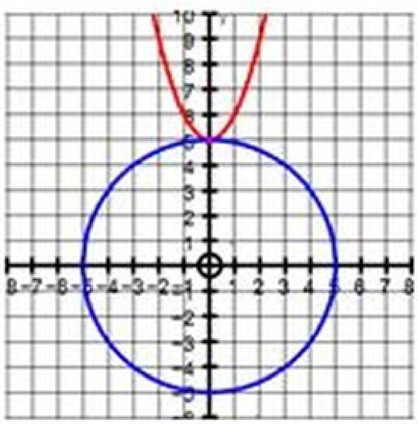
*Consider various cases of the curves* $y = x^2 + c$ *and* $x^2 + y^2 = 25$

*1. When* $c = 6$ *we have* $y = x^2 + 6$ *and* $x^2 + y^2 = 25$



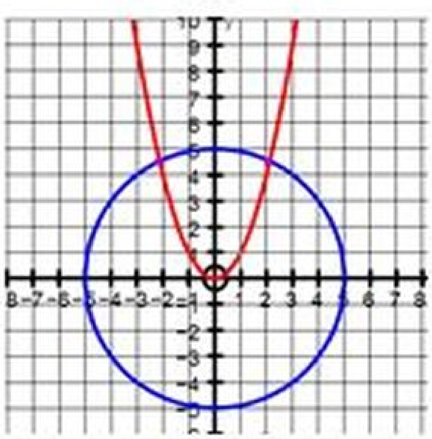
*Here we see that there are apparently no intersection points.*

*2. When* $c = 5$ *we have* $y = x^2 + 5$ *and* $x^2 + y^2 = 25$



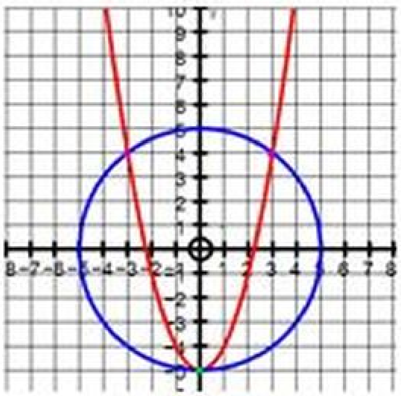
*Here we see that there is apparently 1 intersection point at (0, 5)*

*3. When* $c = 0$ *we have* $y = x^2 + 0$ *and* $x^2 + y^2 = 25$



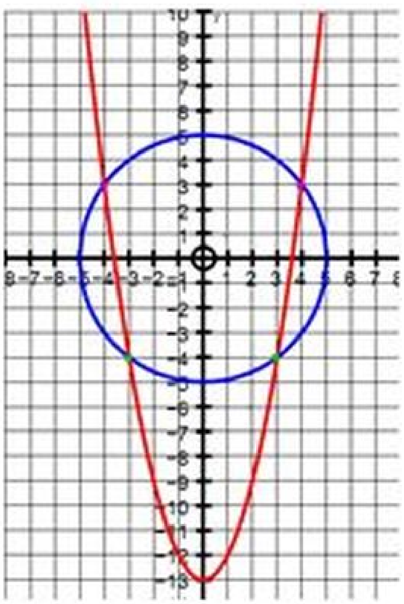
*Here we see that there are apparently 2 intersection points at (2.1, 4.4) and at (–2.1, 4.4)*

4. *When $c = -5$ we have $y = x^2 - 5$ and $x^2 + y^2 = 25$*



*Here we see that there are apparently 3 intersection points at (3, 4) and (– 3, 4) and (0, – 5)*

5. *When $c = -13$ we have $y = x^2 - 13$ and $x^2 + y^2 = 25$*



*Here we see that there are apparently 4 intersection points at (4, 3) and (– 4, 3) and (3, – 4) and (– 3, – 4)*

*Solving these equations algebraically:*

*Subs $y = x^2 - 13$ into $x^2 + y^2 = 25$*

$$x^2 + (x^2 - 13)^2 = 25$$
$$x^2 + x^4 - 25x^2 + 169 = 25$$
$$x^4 - 24x^2 - 144 = 0$$
$$(x^2 - 16)(x^2 - 9) = 0$$
$$(x + 4)(x - 4)(x + 3)(x - 3) = 0$$

*So $x = \pm 4$ and $\pm 3$*
*Intersection points are:*
*(–4, 3), (–3, –4), (3, –4), (4, 3)*

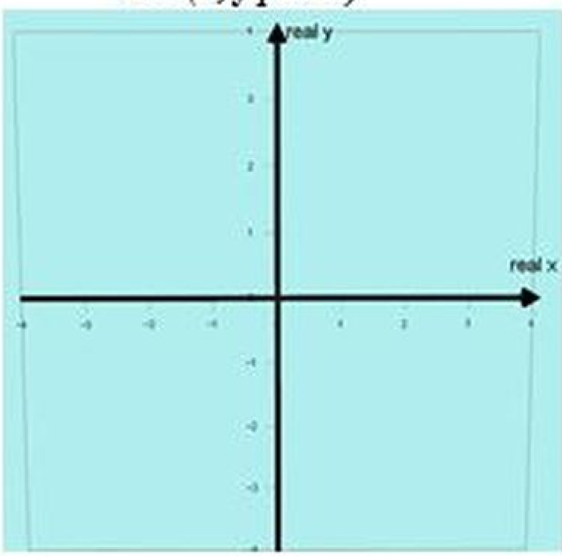
*It occurred to me that if we repeat this algebra for the general case $y = x^2 + c$ we will always get a quartic equation to solve which will always have 4 solutions.*
*Subs $y = x^2 + c$ into $x^2 + y^2 = 25$*

$$x^2 + (x^2 + c)^2 = 25$$
$$x^2 + x^4 + 2cx^2 + c^2 = 25$$
$$x^4 + (2c + 1)x^2 + (c^2 - 25) = 0$$

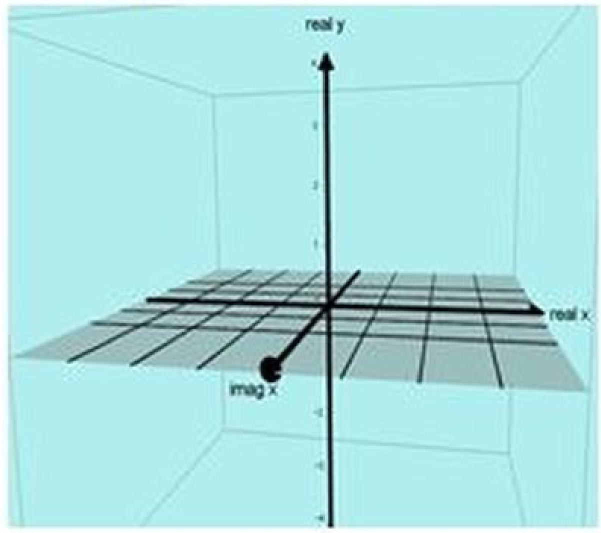
*Logically, if we always should get 4 solutions for different values of c, there must always be 4 intersections which seems contrary to the examples above!*

*The question is "Where are the missing intersections?"*
*The clue of course is in the fact that some of the solutions to the quartic equations are complex numbers.*
*We cannot plot complex points on our ordinary 2Dimensional x, y axes.*
*We need to add an imaginary axis in order to produce a complex x plane with an ordinary y axis sticking up in the middle!*

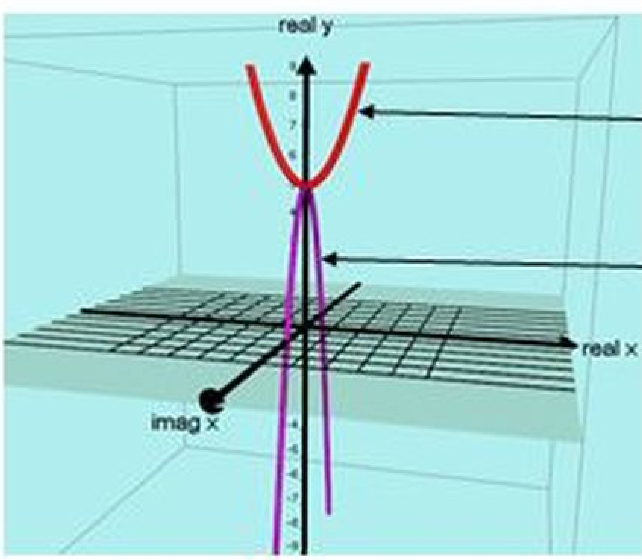
*2D (x, y plane)*




*3D (complex x plane, real y axis)*



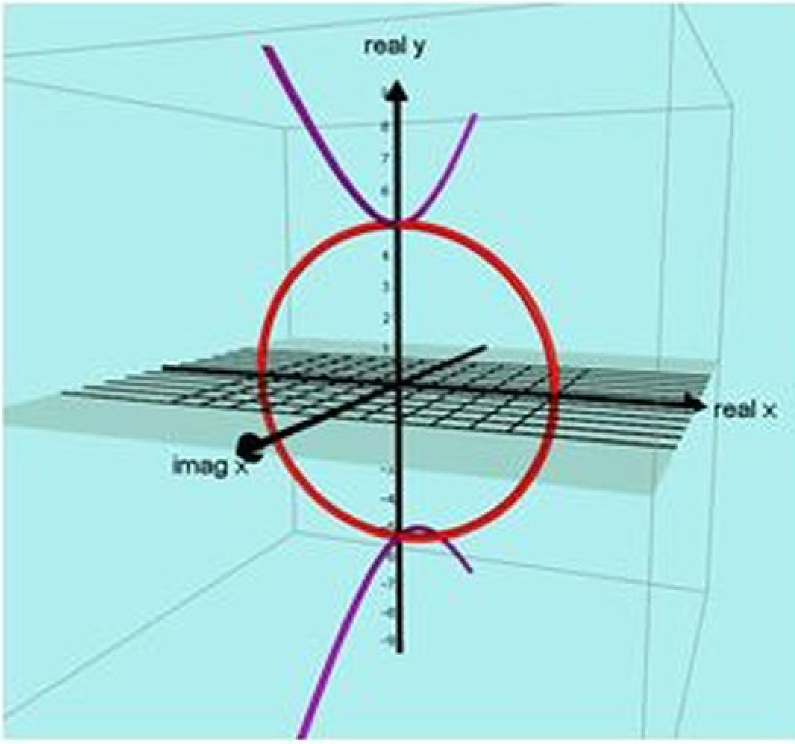

*When we find complex x values which still produce real y values the Parabola has an extra part hanging from its minimum point.*
*(For a detailed explanation of this see [www.phantomgraphs.weebly.com](http://www.phantomgraphs.weebly.com))*




Basic parabola (red)

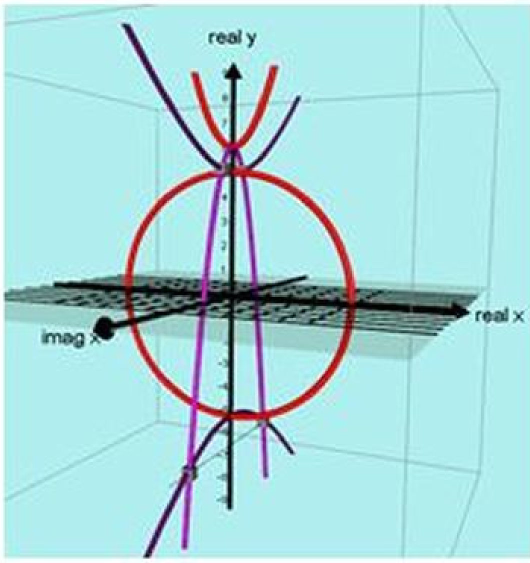
Phantom parabola (purple) is at right angles to the original.

***Similarly, when we find complex x values which still produce real y values the Circle has 2 phantoms which are actually the two halves of a hyperbola!***



***Now I will add both phantom graphs and reconsider the examples at the start of this paper.***

## *1.* $y = x^2 + 6$ *and* $x^2 + y^2 = 25$




*Subs* $y = x^2 + 6$ *into* $x^2 + y^2 = 25$

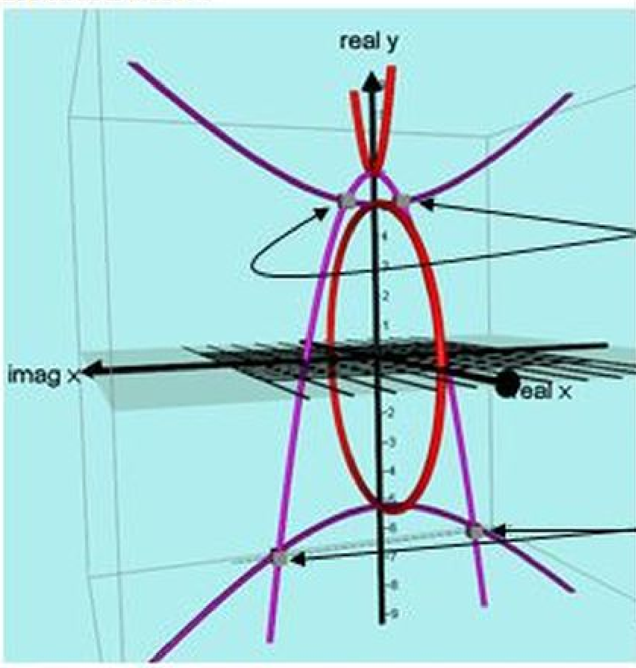
*We get* $x^2 + (x^2 + 6)^2 = 25$

$$x^2 + x^4 + 12x^2 + 36 = 25$$

$$x^4 + 13x^2 + 11 = 0$$

$x = 3.5i, -3.5i, 0.95i, -0.95i$

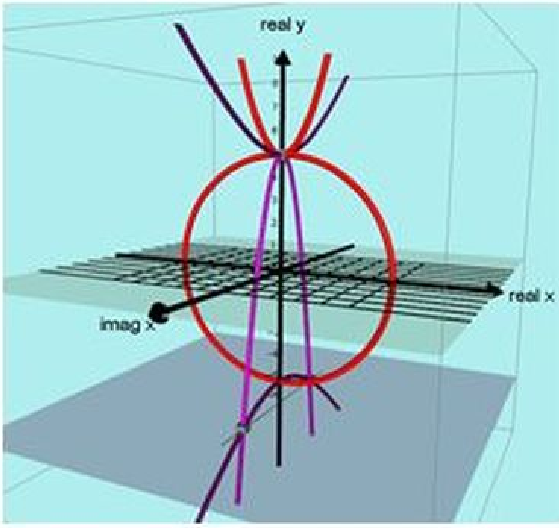
***Rotated view:***



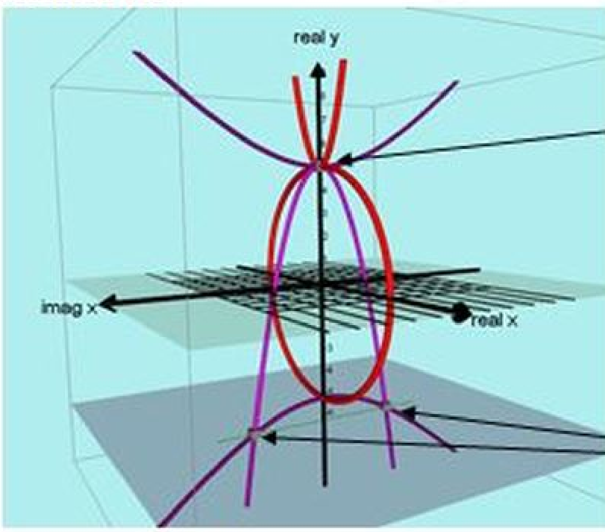

***The parabola's phantom intersects the circle's upper phantom at these 2 points where*** $x = 0.95i$ ***and*** $-0.95i$

***The parabola's phantom intersects the circle's lower phantom at these 2 points where*** $x = 3.5i$ ***and*** $-3.5i$,

## 2. $y = x^2 + 5$ and $x^2 + y^2 = 25$




*Rotated view*




*Subs $y = x^2 + 5$ into $x^2 + y^2 = 25$*

*We get* $x^2 + (x^2 + 5)^2 = 25$

$$x^2 + x^4 + 10x^2 + 25 = 25$$

$$x^4 + 11x^2 = 0$$

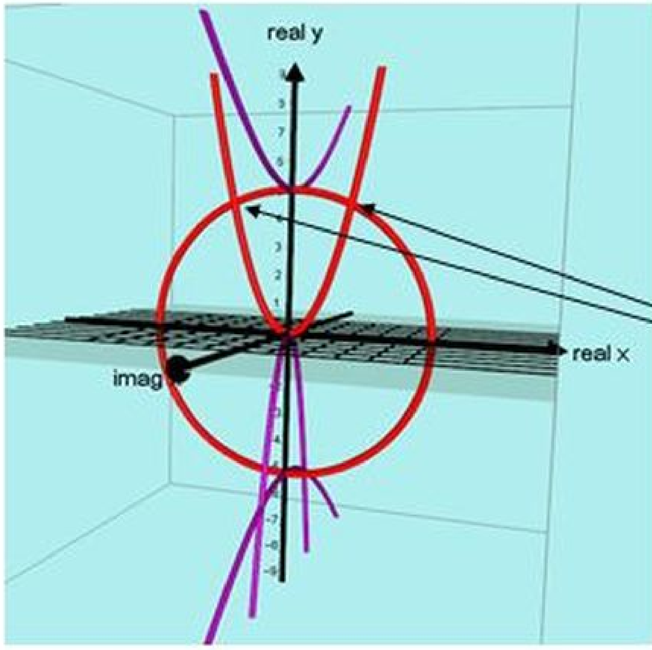
$$x^2(x^2 + 11) = 0$$

*so* $x = 0$ *or* $\pm\sqrt{11}$

*Notice, where $x = 0$ the circle intersects with the parabola and the parabola's phantom also intersects with the circle's upper phantom. So this is a double intersection at (0, 5)*

*The parabola's phantom intersects the circle's lower phantom at these 2 points* $(\sqrt{11}, -6)$ *and* $(-\sqrt{11}, -6)$

### 3. $y = x^2$ and $x^2 + y^2 = 25$




Subs $y = x^2$ into $x^2 + y^2 = 25$

We get $x^2 + (x^2)^2 = 25$

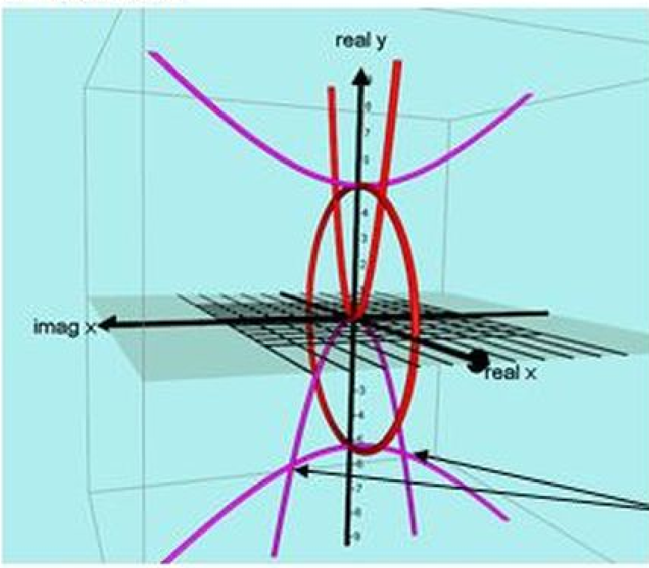
$$x^2 + x^4 = 25$$

$$x^4 + x^2 - 25 = 0$$

$$x = \pm 2.13, \pm 2.35i$$

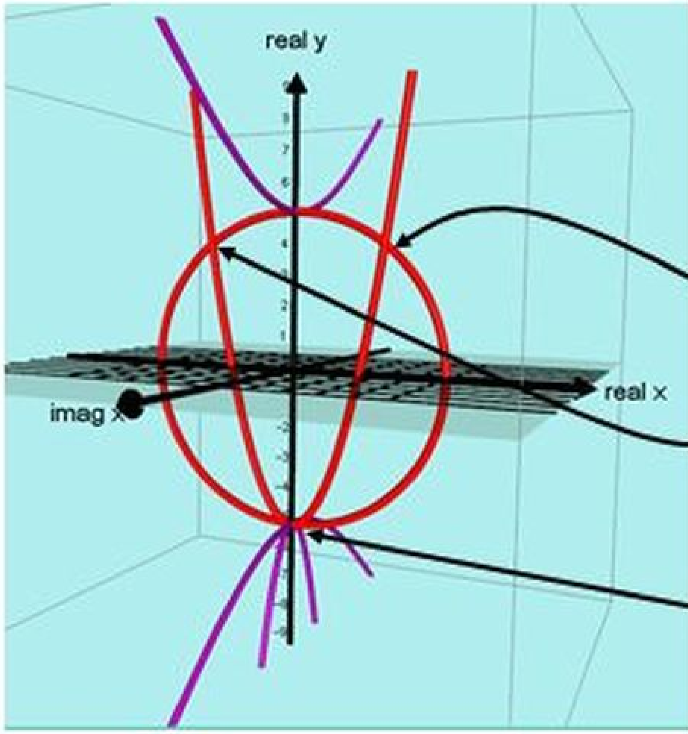
*Notice there are 2 real intersections when $x = \pm 2.13$*

***Rotated view***




*Also notice the parabola's phantom intersects with the lower phantom of the circle at $x = \pm 2.35i$*

***4.*** $y = x^2 - 5$ ***and*** $x^2 + y^2 = 25$




*Subs* $y = x^2 - 5$ *into* $x^2 + y^2 = 25$

*We get* $x^2 + (x^2 - 5)^2 = 25$

$$x^2 + x^4 - 10x^2 + 25 = 25$$

$$x^4 - 9x^2 = 0$$

$$x^2(x^2 - 9) = 0$$

$$x^2(x + 3)(x - 3) = 0$$

$$x = 0, 3 \text{ or } -3$$

*Intersection points are:*

$(-3, 4)$, $(3, 4)$ *and* $(0, -5)$

*Notice the intersections here are all real on the x, y plane.*

*Again there is a "double" solution at (0, –5) because the curves intersect twice at this point. The original red curves intersect and the phantoms also intersect.*

*Also see the screencast video:*

http://screencast.com/t/3bdbGbI1u

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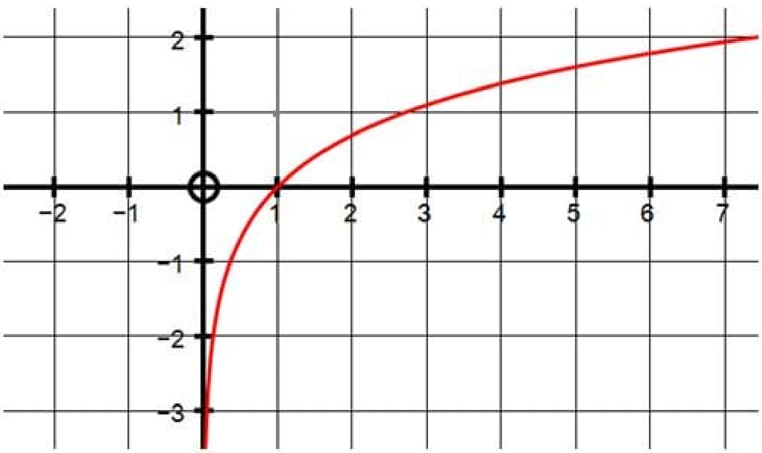
**2017**: **A fascinating extention of the logarithmic graph.**

## The Graph of $y = ln(x)$ for <u>positive</u> AND <u>negative</u> values of $x$.

When we restrict ourselves to the **real numbers**, ln( – 1 ) does not make sense because the graph only seems to exist for $x > 0$




However, if we allow ***complex y values*** we can actually find values of ***ln(x)*** for negative ***x*** values!

We know the three series for $e^x$, $\sin(x)$ *and* $\cos(x)$:

$$\begin{cases} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \\ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \end{cases}$$

So let us consider $e^{i\theta}$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$= \qquad \cos(\theta) \qquad\qquad + \; i\sin(\theta)$$

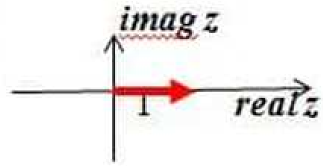
*Or* $cis(\theta) = e^{i\theta}$

If $z = rcis(\theta) = re^{i\theta}$

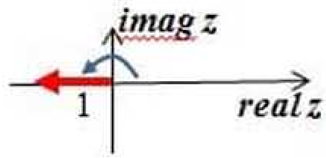
Then $\ln(z) = \ln(re^{i\theta})$
$\qquad\qquad = \ln(r) + \ln(e^{i\theta})$

So $\quad \ln(z) = \ln(r) + i\theta$

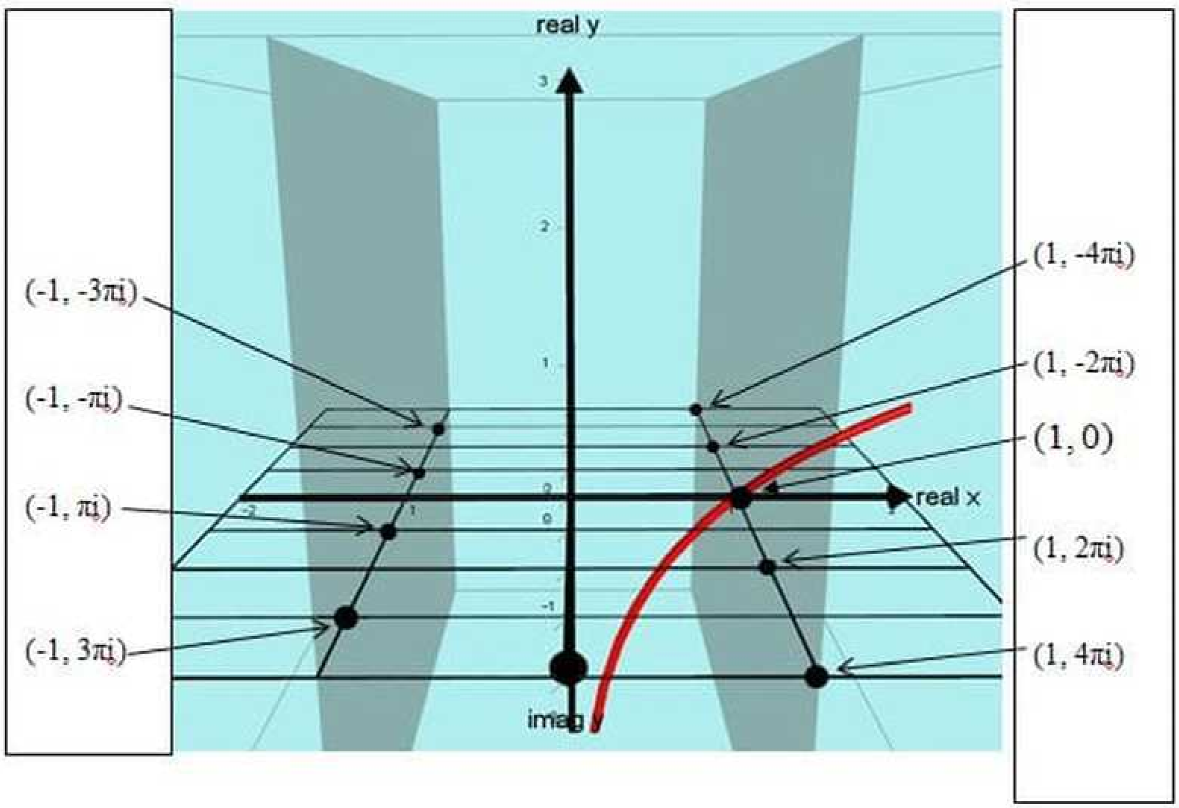
Now this is very interesting because if we let $z = 1 + 0i$
then $r = 1$ and $\theta$ is not just 0 but $2n\pi$ where $n \in$ ***Integers***
so $\ln(1) = 0 + 2n\pi i$
This means that for the graph $y = \ln(x)$
if $x = 1$ then $y$ could be 0 or $\pm 2\pi i$ or $\pm 4\pi i$ or $\pm 6\pi i$ etc



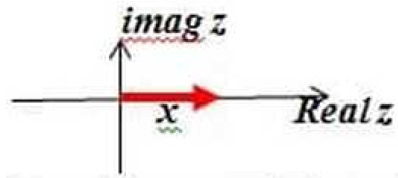

Also if we let $z = -1 + 0i$
then $r = +1$ and $\theta = \pi$
so $\ln(-1) = 0 + (2n + 1)\pi i$
This means that for the graph $y = \ln(x)$
if $x = -1$ then $y$ could be $\pm \pi i$ or $\pm 3\pi i$ or $\pm 5\pi i$ etc



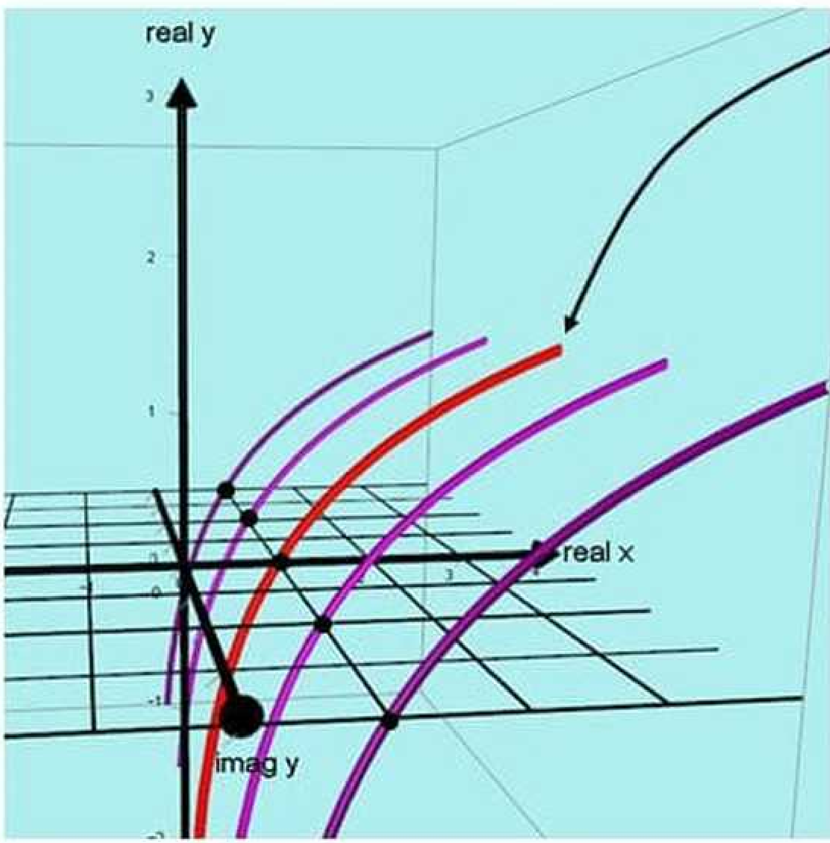

Somehow, the graph of $y = \ln(x)$ is not just the **RED** graph below but it also passes through all the marked points!



***If z is just <u>any positive</u> real number $z = x + 0i$***




***then $r = x$ and $\theta = 0, \pm 2\pi, \pm 4\pi$ etc***
***This means the graph of $y = \ln(x)$ really consists of countless parallel graphs spaced at distances of $2\pi i$ to each other.***
***The equations are $y = \ln(x) + 2n\pi i$***



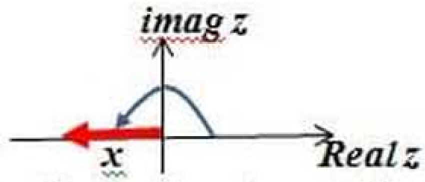

The **RED** graph is the usual
$y = \ln(x)$
It is in the x, y plane and when x = 1 then y = 0

The spacing lines in the imaginary y direction are at intervals of $2\pi$ rads

The PURPLE graphs are all parallel to the red graph at distances of $\pm 2\pi$ and $\pm 4\pi$
When x = 1 then $y = \pm 2\pi i$ and $\pm 4\pi i$

The equations are:
$y = \ln(x) \pm 2\pi i$ and
$y = \ln(x) \pm 4\pi i$

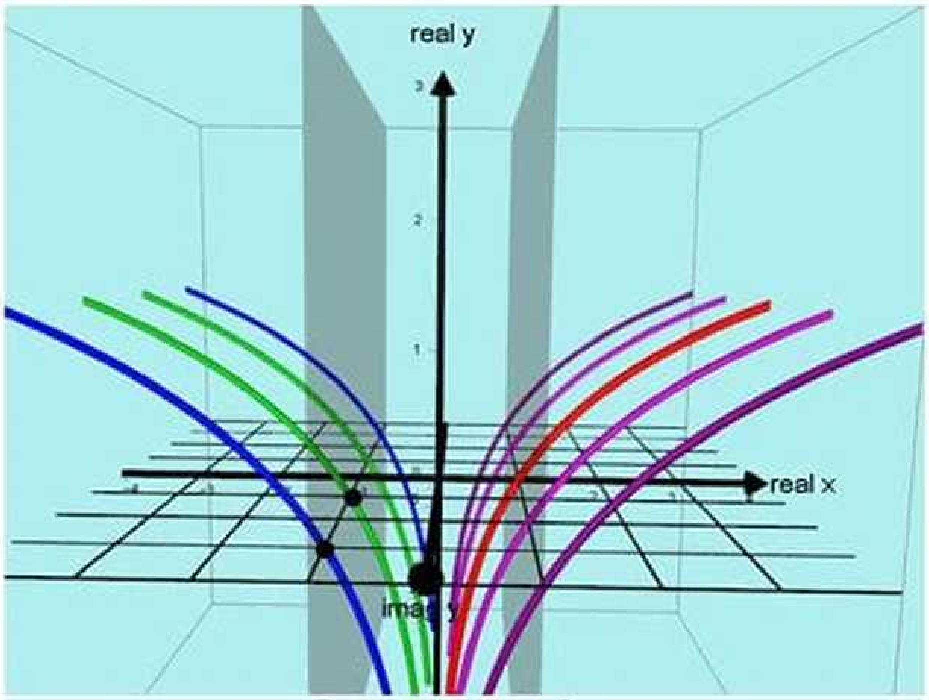
If $z$ is a negative real number $z = -x + 0i$
then $r = |x|$ and $\theta = \pm\pi, \pm 3\pi, \pm 5\pi$ *etc*



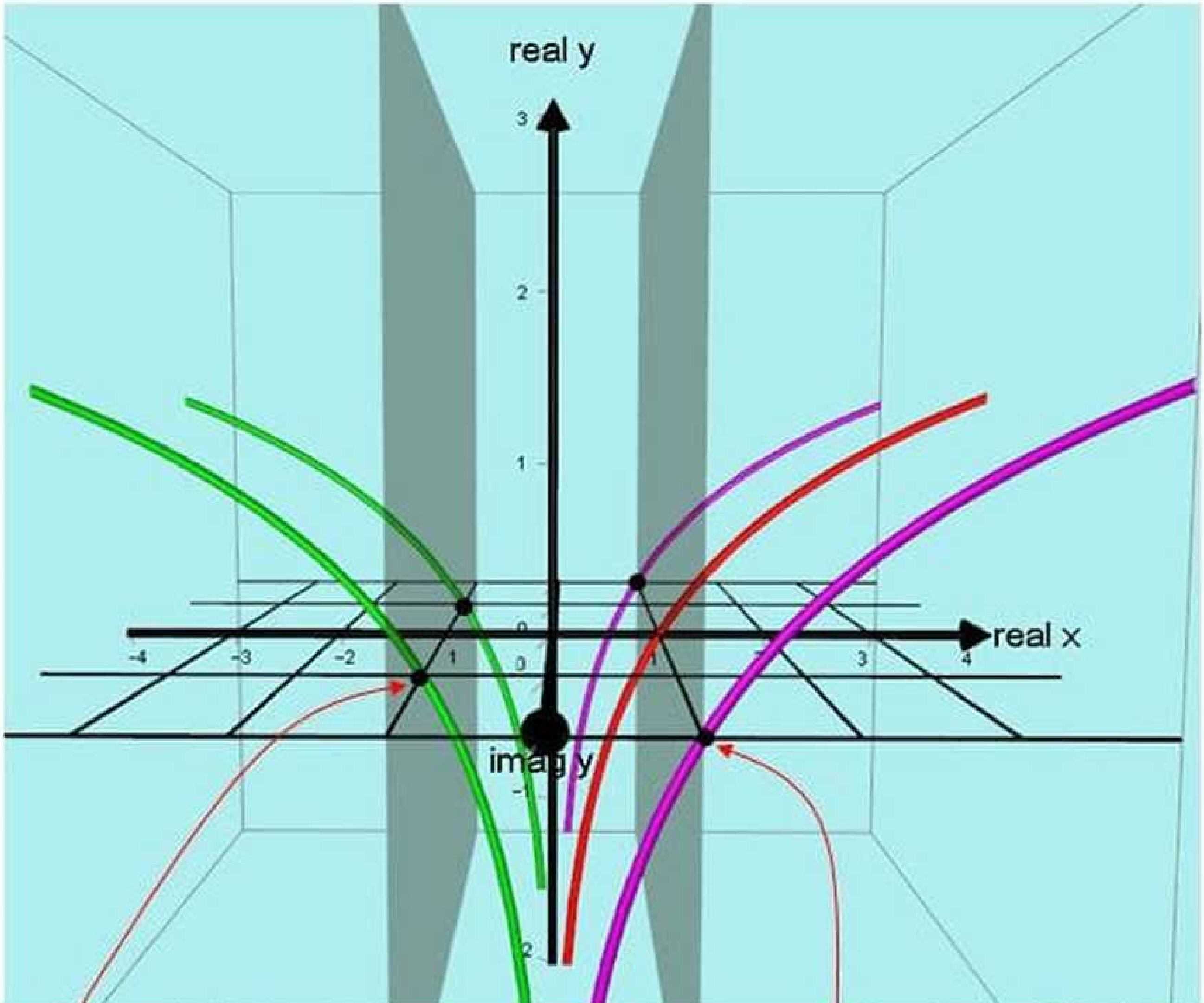

This means the graph of $y = ln(x)$ where $x$ is a negative real number, really consists of countless parallel graphs also spaced at distances of $2\pi$ to each other. The equations are $y = ln|x| + (2n+1)\pi i$

The two GREEN graphs are:
$y = ln|x| \pm \pi i$
When $x = -1$
then $y = \pm \pi i$

The two BLUE graphs are:
$y = ln|x| \pm 3\pi i$
When $x = -1$
then $y = \pm 3\pi i$



For better clarity, I will just include y = $\ln(x) \pm 2\pi i$ for positive $x$ values
and for negative $x$ values I will just include y = $\ln|x| \pm \pi i$



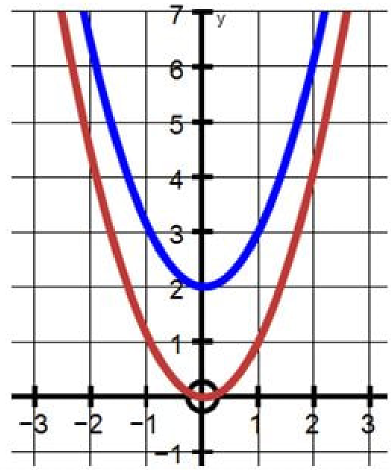

***This point is $(-1, \pi i)$***
***which shows $\ln(-1) = \pi i$***

***This point is $(1, 2\pi i)$***
***which shows $\ln(1) = 2\pi i$***

# DO ALL PARABOLAS HAVE INTERSECTION POINTS?

Firstly, let's look at a fairly obvious case:

$y = x^2$ and $y = x^2 + 2$
The intersection would be when:
$x^2 + 2 = x^2$
but subtracting $x^2$ from both sides
we get: $2 = 0$
Obviously $x^2 + 2$ can never equal $x^2$
And it is clear the graphs do not intersect.

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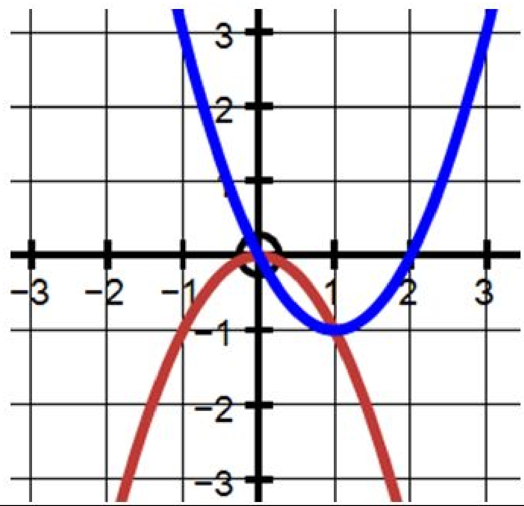
An ordinary case where they do intersect would be for these two parabolas:
$y = -x^2$ and $y = x^2 - 2x$




The intersection points would be when:
$x^2 - 2x = -x^2$
$2x^2 - 2x = 0$
$2x(x - 1) = 0$
$x = 0$ and $x = 1$
$y = 0$ and $y = -1$

Intersection points are $(0, 0)$ and $(1, -1)$

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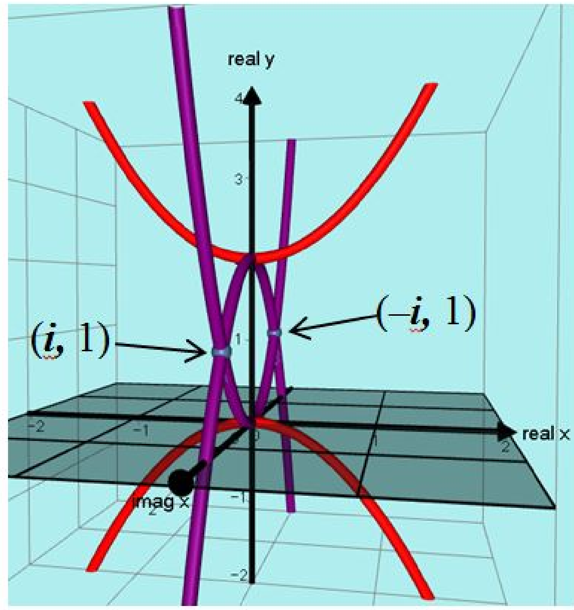
Probably a very surprising case is the intersection of $y = -x^2$ and $y = x^2 + 2$

On solving the equations we get:
$x^2 + 2 = -x^2$
so $2x^2 = -2$
and $x^2 = -1$
so $x = i$ and $-i$

If we allow **complex x** values but just keep **real y** values then the ***phantom*** graphs show there are two intersection points!
The two basic graphs are **RED** and their phantoms are **PURPLE**.
The intersection points are $(i, 1)$ and $(-i, 1)$

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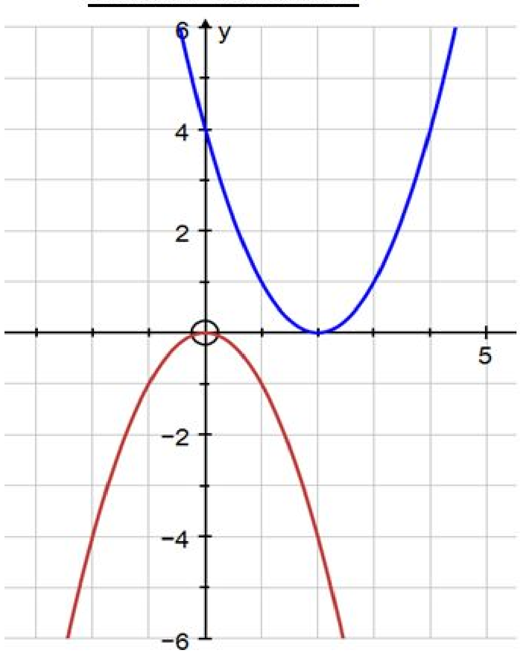
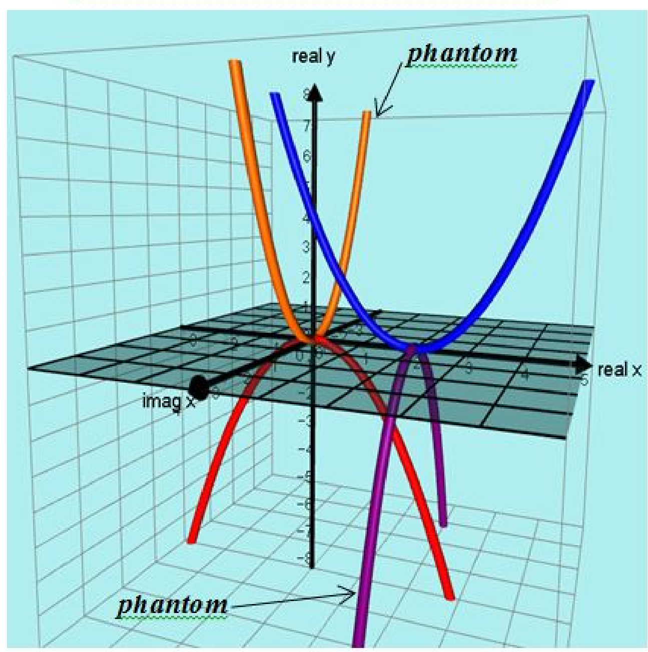
However, consider these two parabolas: $y = -x^2$ *and* $y = (x-2)^2$

<u>BASIC GRAPHS</u>.




<u>GRAPHS WITH *PHANTOMS*.</u>




Clearly the **basic graphs do not intersect** and **neither do their phantoms** but if we do the algebra, we get an answer!

The intersection of $y = -x^2$ and $y = (x-2)^2$ would be when $(x-2)^2 = -x^2$

$$x^2 - 4x + 4 = -x^2$$

$$2x^2 - 4x + 4 = 0$$

$$x^2 - 2x + 2 = 0$$

$$x^2 - 2x + 1 = -2 + 1 \text{ (completing the square)}$$

$$(x-1)^2 = -1$$

$$x - 1 = \pm i$$

$$x = 1 + i \text{ and } x = 1 - i$$

$$y = -(1+i)^2 \text{ and } y = -(1-i)^2$$
$$y = -2i \text{ and } y = 2i$$

The intersection points are $(1 + i, -2i)$ and $(1 - i, 2i)$
(These two points do satisfy each of the parabolic equations!)

This is different from the normal *Phantom Graph Theory*!

In the normal *Phantom Graph Theory* we have **complex x values**, but ONLY **real y values**.
The above intersection points have both coordinates involving imaginary parts.

The consequence of this is that we would need **4 axes** to plot such points!
They would be: Real and Imaginary x axes and Real and Imaginary y axes.
Unfortunately, this requires FOUR dimensions!
Even Autograph cannot cope with this!
But I have shown that it is possible to find 4D intersection points for those parabolas that we did not think would intersect at all!

**CONTACT PHILIP LLOYD (Specialist Calculus Teacher) by email:**
**philiplloyd1@gmail.com**

**IF YOU DECIDE TO USE ANY OF MY RESOURCES I WOULD BE PLEASED IF YOU COULD SEND ME AN EMAIL.**
**I would appreciate any feedback.**

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